Dispersing Fermi·Ulam Models

Jacopo De Simoi · University of Toronto

joint work with
Dmitry Dolgopyat · UMD

The Dynamical Systems, Ergodic Theory, and Probability conference dedicated to the memory of Nikolai Chernov

UAB 20·05·2015

Fermi·Ulam model

One-dimensional mechanical model

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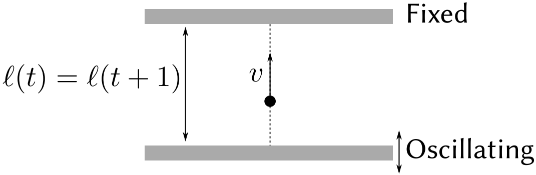

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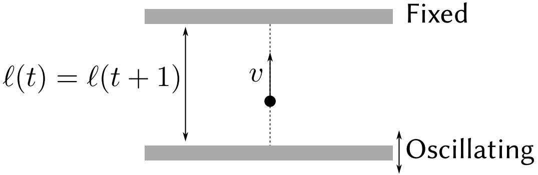

Elastic collisions with the walls + free motion in between
Energy is **not** preserved at collisions

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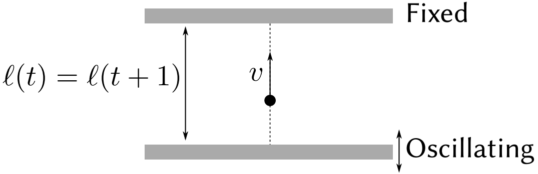

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Goal: describe the long term energy distribution.

Fermi·Ulam model

Conjecture (Fermi·Ulam^{1951})

"One expects that, after sufficiently long times, the average velocity of the point will become very large [...]. The tendency towards equipartition of energy would imply this."

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If $\ell(t)$ is analytic, all trajectories have bounded energy

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If $\ell(t) \in C^{4+\epsilon}$*, all trajectories have bounded energy*

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Theorem (Zharnitsky[1999])

There exist $\ell(t) \in C^0$ so that some trajectories have unbounded energy

The collision map

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time of collision	t
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$$M = \mathbb{T} \times \mathbb{R}^+ \qquad f : (t_n, w_n) \mapsto (t_{n+1}, w_{n+1})$$

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$$f(t_n, w_n) = (t_n + \delta t(t_n, w_n), w + \dot{\ell}(t_n) - \dot{\ell}(t_{n+1})),$$

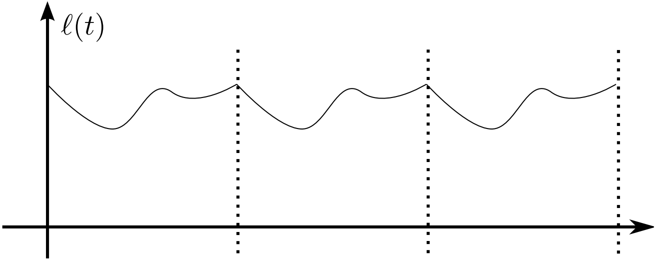
where for $w > \|\dot{\ell}\|$:

$$\delta t(t, w) = \frac{\ell(t) + \ell(t + \delta t(t, w))}{w - \dot{\ell}(t)} = \mathcal{O}(w^{-1})$$

for $w \to \infty$, f is close to integrable $(t, w) \mapsto (t + w^{-1}, w)$

Our general standing assumptions

Recall: $\ell(t)$ is distance between the two walls
$\ell(t) \in C^5[0,1]$ strictly positive, $\ell(0) = \ell(1)$, $\dot{\ell}(0) \neq \dot{\ell}(1)$.



Early results: asymptotic normal form

Define *fundamental parameter* [$\sim$ discontinuity]:

$$\Delta_\ell = \left[\ell(0)\int_0^1 \ell(s)^{-2}\mathrm{d}s\right]\cdot\left(\dot\ell(0)-\dot\ell(1)\right)$$

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Theorem (—·Dolgopyat[2012])

There exists $R \simeq [0,1]\times\mathbb{R}^+$, $R \subset M$ *and coordinates* (θ, I) *on* R *with* $I \sim v$ *such that the first return map* F *of* f *on* R *is a* $\mathcal{O}_5(I^{-1})$*-perturbation of* $\hat F : (\theta, I) \mapsto (\bar\theta, \bar I)$ *where*

$$\bar\theta = \theta - I \mod 1 \qquad\qquad \bar I = I + \Delta_\ell(\bar\theta - 1/2)$$

The piecewise linear Standard Map I

$$\bar{\theta} = \theta - I \mod 1 \qquad \bar{I} = I + \Delta_\ell(\bar{\theta} - 1/2)$$

$\hat{F}$ is a piecewise affine map: $\mathrm{Tr}\, \mathrm{d}\hat{F} \equiv 2 - \Delta_\ell$.

Corollary (of the Normal Form)

If $\Delta_\ell \in (0, 4)$ *the dynamics of the FUM is* asymptotically elliptic*; the dynamics can be described [up to* $\mathcal{O}(I^{-1})$*] as a piecewise isometry.*

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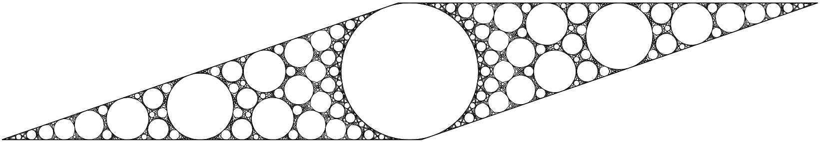

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Moreover, let $\mathcal{E} = \{(t_0, w_0)$ s.t. $\lim_{n\to\infty} w_n = \infty\}$, then:

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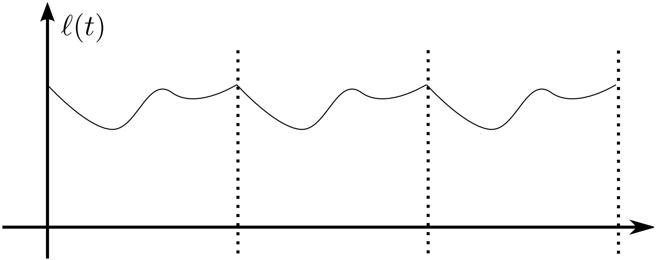

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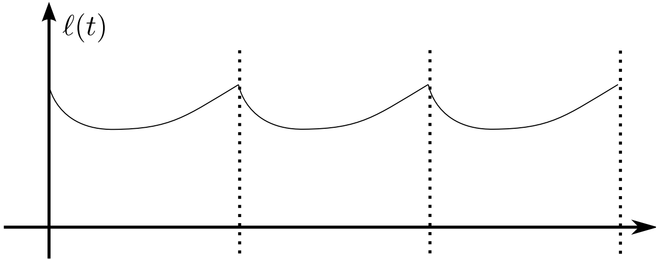
+ Convexity assumption: $\ddot{\ell} \geq \kappa > 0$



Dispersing Fermi·Ulam models

Definition

A FUM is *dispersing* if $\ddot{\ell} \geq \kappa > 0$ (in particular $\Delta_\ell < 0$)

Theorem (—·Dolgopyat[2015])

There exists a discrete set E s.t. if $\Delta_\ell \notin E$, the dynamics f of a dispersing FUM is ergodic.

e.g. the energy of a.e. trajectory can infinitely often be arbitrarily large and arbitrarily small.

Hints of the proof: hyperbolicity

Lemma

The dynamics f of a dispersing FUM is cone-hyperbolic, i.e

$$\exists\, \mathcal{K}^{u,s}(x) \subset T_xM \text{ s.t. } Df\mathcal{K}^u(x) \subset \mathcal{K}^u_{f(x)}, Df^{-1}\mathcal{K}^s(x) \subset \mathcal{K}^s_{f^{-1}(x)}$$

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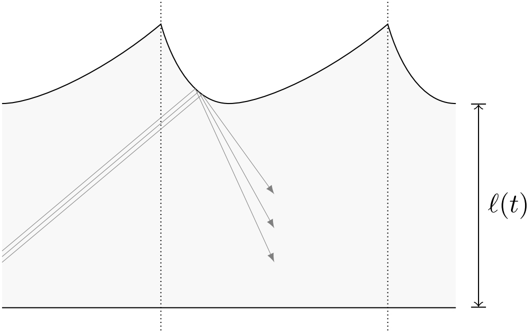
Proof: f preserves the positive cone in (suitable) Jacobi coords $\square$



Hints of the proof: hyperbolicity

However:

$$\min_{v \in \mathcal{K}^u(x)} \frac{\|Dfv\|_p}{\|v\|_p} \geq 1 + 2\tau(x)\kappa/w$$

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- Large velocities
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Lemma

The induced map $\hat{f} : \hat{M} \to \hat{M}$ is uniformly hyperbolic.

Hints of the proof: from hyperbolicity to ergodicity

$$\text{Ergodicity} = \text{local ergodicity (via Hopf argument)} \\ + \text{combinatorial information}$$

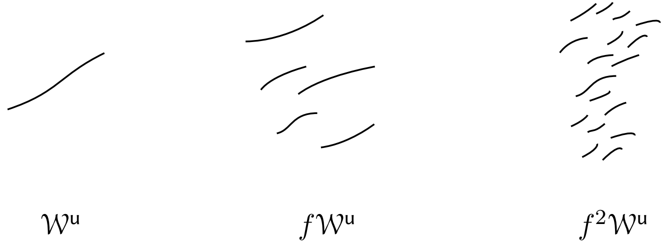
Ingredients

- Growth Lemma
- Distortion bounds for the (un)stable manifolds
- Absolute continuity of (un)stable manifolds

Our system has **infinite volume**!

Hints of the proof: the Growth Lemma

Unstable manifolds are *expanded* by hyperbolicity, but *fragmented* by singularities



Growth Lemma

Let $l_n(x) = |$ c.c. of $f^n\mathcal{W} \ni f^n x|$: if $n > C|\log|\mathcal{W}||$,

$$\mathbb{P}_{\mathcal{W}}(l_n(x) < \varepsilon) < C\varepsilon$$

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Growth Lemma for planar billiards with corner points, finite horizon

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Theorem (—·P.I. Tóth[2013,WIP])

Growth Lemma for planar billiards with corner points, infinite horizon

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Hints of the proof: the Growth Lemma II

Fact

Subexponential complexity bounds on compact sets and at infinity (i.e. for the normal form) does not necessarily imply the Growth Lemma

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Additional assumption

Bounded complexity for the normal form (holds for $\Delta_\ell \notin E$)

$\Rightarrow$ Growth Lemma

Thank you!