Kolya's legacy in dynamics and mathematical physics

Overview

Disclaimers.

1. Discussion limited to dynamics and mathematical physics (not statistics)
2. Certain historical perspective may be lacking

Overview

Disclaimers.

1. Discussion limited to dynamics and mathematical physics (not statistics)
2. Certain historical perspective may be lacking

Plan of the talk.

1. Kolya's work on billiards
2. Kolya's contribution beyond billiards
3. Applications of Kolya's results to mathematical physics

Kolya and Chaotic Billiards

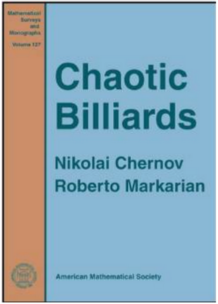
Kolya was a recognized leader in the field of chaotic billiards

• developed (jointly with Sinai) a standard approach to proving ergodicity in hyperbolic systems with singularities

• obtained (with various collaborators) a detailed description of statistical properties of dispersing billiards

• put a lot of effort in simplifying and generalizing original proofs

• a book Chaotic billiards of Chernov and Markarian gives an overview of the field and explains key ideas and techniques



Chernov-Sinai approach to ergodicity of systems with singularities (1987)

Synopsis (based on Kolya's 1993 paper).

Setting: systems with singularities have non-zero Lyapunov exponents almost everywhere.

E^u, E^s-stable and unstable (Oseledec) subspaces.

Adapted metric:

$$||DT(v)|| \geq ||v|| \text{ if } v \in E^u, \quad ||DT^{-1}(v)|| \geq ||v|| \text{ if } v \in E^s.$$

x is u-essential if $\forall \Lambda \; \exists V \ni x$ and $n > 0$:

$$||dT^n(v)|| \geq \Lambda ||v|| \text{ for all } y \in V, v \in E^u(y).$$

x is s-essential if it is u-essential for T^{-1}.

$\bar{x}$ is sufficient if there is $x = T^k \bar{x}$, $V \ni x$, $\Lambda > 1$ and $n > 0$: in V

$$||dT^n(v)|| \geq \Lambda ||v||, \quad ||dT^n(u)|| \leq \Lambda^{-1} ||u||, \quad v \in E^u(y), u \in E^s(y).$$

Chernov-Sinai conditions

1. Double singularites have codimension at least 2
2. Oseledec decomposition is continuous at sufficient points
3. Small neighbourhood of singularities has small measure
4. (Ansatz) Almost all points on singularities are essential
5. Singularities are transversal to stable and unstable manifolds almost everywhere

Theorem Properties (1)-(5) imply local ergodicity near sufficient points.

Chernov-Sinai conditions

1. Double singularites have codimension at least 2
2. Oseledec decomposition is continuous at sufficient points
3. Small neighbourhood of singularities has small measure
4. (Ansatz) Almost all points on singularities are essential
5. Singularities are transversal to stable and unstable manifolds almost everywhere

Theorem Properties (1)-(5) imply local ergodicity near sufficient points.

- Separates difficulties comping from zero exponetns (insuffciency) and singularites
- Based on finitely verifable conditions

Chernov-Sinai conditions

1. Double singularites have codimension at least 2
2. Oseledec decomposition is continuous at sufficient points
3. Small neighbourhood of singularities has small measure
4. (Ansatz) Almost all points on singularities are essential
5. Singularities are transversal to stable and unstable manifolds almost everywhere

Theorem Properties (1)-(5) imply local ergodicity near sufficient points.

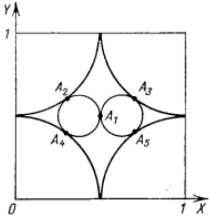
Moreover each ergodic component $E = E_0 \cup E_1 \cup E_{m-1}$ where $T : E_j \to E_{j+1 \bmod 1}$ and (T^m, E) is a K-system.

Chernov–Haskell (1996): (T^m, E) is a Bernoulli.

Chernov-Sinai (1987) results

In the context of hard sphere gas sufficiency means rich collision combinatorcs.

(I) Three discs on $\mathbb{T}^2$: global ergodicity in high density regime.



(II) Arbitrary many discs on $\mathbb{T}^2$: local ergodicity on positive (large?) measure set in a low density regime.

(III+) Many applications to systems with elastic collisions (new results at this conference).

Chernov-Sinai Theory beyond billiards

Setting: There is a well defined set $\mathcal{C}$ where hyperbolicity fails. E.g.: In billiards, invariant manifolds are fractured near singularities.
Key idea: If most points on the trajectory of $\mathcal{C}$ are hyperbolic then one can recover lost hyperbolicity having good stochastic properties for almost all orbits.
Examples

- Maps with critical points (where the map is not a local diffeomorphism)
- Maps with homoclinic tangencies

Question. Present a version of Chernov-Sinai theory where the continuity of Oseledec decomposition is replaced by a weaker condition.

Work of Bunimovich-Chernov-Sinai on dispersing billiards

A large class of dispersive billiards is considered including finite and infinite horizon gas and billiards with corners.

- Suitable sympolic dynamics is constucted
- Stretched exponential decay of correlations are proven
- Central Limit Theorem is obtained
- Introduced a lot of technical tools including Markov sieves

A (n, N)-Markov sieve with $n \sim N^\gamma$ for $\gamma < 1$ is a partition $\mathcal{M} = \bigcup_{i=0}^{I} A_i$ such that

- $\text{diam}(A_i) < e^{-n}$ for $i > 0$
- $\text{mes}(A_0) \leq Ne^{-n}$
- The process $i_n(x)$ where $T^n x \in A_{i_n}$ is well approximated by Markov process if $T^n x \notin A_0$
- The set $A_1 \dots A_I$ enjoys good mixing properties (e.g. Doeblin condition) at scale n.

Applications of Markov sieves

- Bunimovich-Chernov-Sinai (1991) proved that (n, N) sieves exist for all N and deduced stretched exponential mixing and CLT
- the same program realized by Chernov for contact Anosov flows (1998) and billiard flows (2007)
- slow-fast system (e.g. the work of Chernov and Dolgopyat on Brownian Motion (2009))
- finite time chaos

Mixing rates in billiards

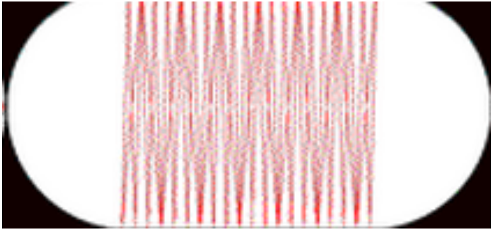
• Young (1998): exponetial mixing for finite horizon Lorentz gas
• Chernov (1999): exponetial mixing for infinite horizon Lorentz gas
Let Δ_n be the displacement of the Lorentz particle between n-th and $n+1$-st collision. Note that $\mathbb{E}(|\Delta_0|^2) = \infty$.
Lemma (Chernov-Dolgopyat (2009)) $|\mathbb{E}(\Delta_0 \Delta_n)| \leq K\theta^n$ for $n > 0$.
• Chernov (1999): exponential mixing for billiards with corners (under complexity assumption removed by de Simoi-Toth (2014))
A key role is played by an n-step expansion estimate

$$\sum_{\text{components of } T^n W} \lambda_i^{-1} \leq \eta < 1$$

emphasized by Kolya.

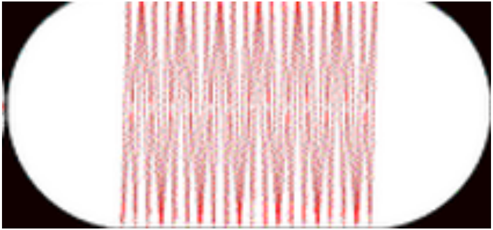
Slowly mixing billiards.

• In many semidispersing and defocusing billiards the mixing due to slow divergence of nearby trajectories.



Slowly mixing billiards.

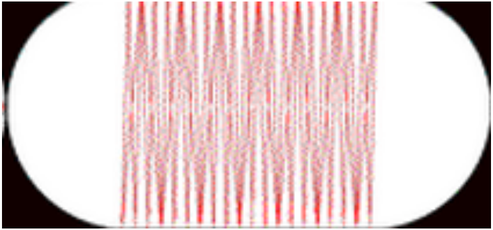
• In many semidispersing and defocusing billiards the mixing due to slow divergence of nearby trajectories.



• A general approach to slowly mixing systems was proposed by Young based on Young tower with well controlled return time.

Slowly mixing billiards.

• In many semidispersing and defocusing billiards the mixing due to slow divergence of nearby trajectories.



• A general approach to slowly mixing systems was proposed by Young based on Young tower with well controlled return time.
• If the set where hyperbolicity fails is complicated it is often difficult to obtain sharp bounds on the tail of return time.

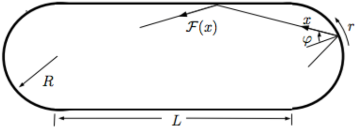
Slowly mixing billiards.

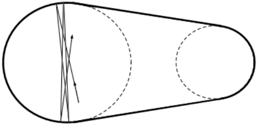
Markarian (2004) developed and Chernov-Zhang (2005–2008) refined the following procedure

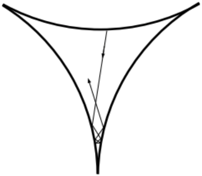
1. Find a subset $M \subset \mathcal{M}$ such that an induced map has good mixing properties: (e.g. Young tower with exponentially decaying return times)
2. Study the return time τ from $\mathcal{M}$ to M
3. Relate the properties of τ and the induced map T_M to the mixing properties of the original map T.

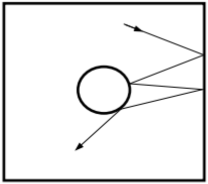
Slowly mixing billiards.

Chernov-Zhang obtained optimal polynomial bounds for many clasiscal examples inluding Bunimovich stadia, billiards with cusps, rectangular billiards with holes, flowers etc.









Mixing-1.

Chernov (1992) proves exponential mixing for piecewise linear toral automorphisms.

Method: Renewal Theory.
Since the map is piecewise linear the Lebesgue measure induces a Markov measure for the symbolic system, so renewal theory for Markov chains could be used.

Later developments: operator valued renewal theory by Sarig (2002), Gouezel (2004) . . .
Current research deals with infinite measures systems, flows etc.

One approach to proving renewal theorems is via Tauberian theorems for Fourier transform. In dynamics, transfer operators are needed (cf. Young (1998)).

Mixing-2.

In 1999 paper Kolya proved mixing for a wide class of multidimensional hyperbolic systems with singularities including piecewise linear maps.

Important new ingredient: controlling the sizes of invariant manifolds in high dimesnions.

Currently relations between distribution of sizes of Pesin manifolds and ergodic and mixing properties of the system is an active area of research.

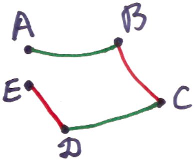
Some contributors:

Alves, Bonatti, Viana, Burns, Dolgopyat, Pesin, Climenhaga . . .

Mixing-3 (flows).

Phase space of a hyperbolic flows locally is a section with hyperbolic map $\times$ flow direction. Mixing inside the section was well understood.

Kolya's insight: Contact structure could be used to effectively control mixing in flow direction.



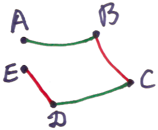
$$t(A) - t(E) = \int_E^A \lambda = \int_{ABCDE} \lambda$$

$$= \int_{BCDE} d\lambda + \int_{ABE} d\lambda = \text{Area}(BCDE).$$

Using Markov sieves Kolya proved stretched exponential bounds for geodesic flows (1998) and billiard flows (2007)

Mixing-3 (flows).

Kolya's insight: Contact structure could be used to effectively control mixing in flow direction.



$$t(A) - t(E) = \int_E^A \lambda = \int_{ABCDE} \lambda$$

$$= \int_{BCDE} d\lambda + \int_{ABE} d\lambda = \text{Area}(BCDE).$$

Using Markov sieves Kolya proved stretched exponential bounds for geodesic flows (1998) and billiard flows (2007)

Currently there are sophisticated methods to exploit Kolya's insight including exotic spaces constructed by Liverani, Tsujii, Faure, Sjostrand . . . There are very precise results on contact Anosov flows (exponential mixing, location of resonances etc).

Question: Non-contact flows?

Anosov diffeomorphisms with holes

Motivation: Open billiards.

Question: Describe the measure and distribution of the set of orbits which do not exit the system after n collisions.

Chernov-Markarian-Troubetzkoy (1997-1998) studied Anosov diffeos on $\mathbb{T}^2$ with holes.

In previous works Pianigiani-Yorke (1979), Lopes-Markarian (1996) the surviving set had a simple Cantor structure.

The result: For small hole size the authors construct conditionally invariant SRB measures

$$T_*\mu_u = \lambda_u \mu_u.$$

λ_u controls the measure of the surviving set and μ_u controls its distribution.

Anosov diffeomorphisms with holes

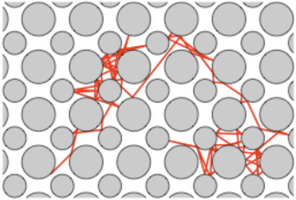
Question: Describe the measure and distribution of the set of orbits which do not exit the system after n collisions.

Chernov-Markarian-Troubetzkoy (1997-1998) studied Anosov diffeos on $\mathbb{T}^2$ with holes.
The result: For small hole size the authors construct conditionally invariant SRB measures $T_*\mu_u = \lambda_u \mu_u$.
λ_u controls the measure, μ_u controls the distribution.

Currently-an active area of research. Some papers:
Bunimovich-Dettmann
Bunimovich-Yurchenko
Demers-Wright-Young
Dyatlov-Guillarmou
Dolgopyat-Nandori ...

Question: Open billiards

Ohm's Law for Lorentz gas.



Theorem (Bunimovich-Chernov-Sinai (1991)) $\frac{q(t)}{\sqrt{t}} \Rightarrow \mathcal{N}(0, D^2)$.

Thermostated Lorentz gas:
$\ddot{q} = E - \frac{\langle \dot{q}, E \rangle}{|\dot{q}|^2} \dot{q}$.

Theorem (Chernov-Eyink-Lebowitz-Sinai (1993))

$$\lim_{t \to \infty} \frac{q(t)}{t} = J(E) \text{ where}$$

$$J(E) = CE + o(|E|) \text{ Ohm's Law and}$$

$$C = \frac{1}{2} D^2 \text{ Einstein relation}$$

Ohm's Law for Lorentz gas.

Theorem (Chernov-Eyink-Lebowitz-Sinai (1993))

$$\lim_{t\to\infty} \frac{q(t)}{t} = J(E) \text{ where } J(E) = \frac{1}{2} D^2 E + o(|E|).$$

$J(E) = \int \Delta d\mu_E$ where Δ is free path vector and μ_E is the physical measure.

For uniformly hyperbolic systems physical measure is weakly smooth ((Katok-Knieper-Pollicott-Weiss (1989))

$J(E)$ is beleived to be **non-smooth** in general for $E \neq 0$ (Bonetto-Daems-Lebowitz (1999))

Ohm's Law for Lorentz gas.

Theorem (Chernov-Eyink-Lebowitz-Sinai (1993))

$$\lim_{t\to\infty}\frac{q(t)}{t} = J(E) \text{ where } J(E) = \frac{1}{2}D^2E + o(|E|).$$

$J(E) = \int \Delta d\mu_E$ where Δ is free path vector and μ_E is the physical measure.

For uniformly hyperbolic systems physical measure is weakly smooth ((Katok-Knieper-Pollicott-Weiss (1989))

$J(E)$ is beleived to be **non-smooth** in general for $E \neq 0$ (Bonetto-Daems-Lebowitz (1999))

Key feature: μ_0 is smooth.

Smoothness of physical measures.

$$\mu_E(A) - \mu_0(A) = \lim_{N\to\infty} \left[\mu_0(A(f_E^N x)) - \mu_0(A(x))\right] =$$

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \left[\mu_0(A(f_E^{k+1} x)) - \mu_0(A(f_E^k x))\right].$$

$$\mu_0(A(f_E^k x)) = \mu_0(A(f_E^k f_0 x)) = \mu_0 A(f_E^{k+1} \delta_E^{-1} x)$$

$$= \int A(f_E^{k+1} y) \frac{d\mu_0(x)}{d\mu_0(y)} d\mu_0(y)$$

where $f_E = f_0 \delta_E x$, $y = \delta_E^{-1} x$.

If μ_0 is not smooth, then the last change of variables is singular unless one allows a mistake in the stable direction, so the situation is much more complicated and there are many open problems.

Einstein relation.

Theorem (Chernov-Eyink-Lebowitz-Sinai (1993))

$$\lim_{t\to\infty} \frac{q(t)}{t} = J(E) \text{ where } J(E) = \frac{1}{2} D^2 E + o(|E|).$$

Consider Lorentz gas without thermostate:

$$\ddot{q} = \varepsilon E, \qquad \frac{|\dot{q}|^2}{2} - \varepsilon\langle q, E\rangle = H.$$

It is a slow-fast system where the slow variable is kinetic energy and fast variables form thermostatted Lorentz gas.

(Chernov-Dolgopyat (2009)) derive Einstein relation from scaling properties of the system (scaling time$\leftrightarrow$ scaling velocity and force) and invariance of the Lioville measure for the full system.

Lorentz gas summury

Finite horizon:

Free motion: (Bunimovich-Chernov-Sinai (1991))
$\frac{q(t)}{\sqrt{t}} \Rightarrow \mathcal{N}(0, D^2)$.

Field with thermostate: (Chernov-Eyink-Lebowitz-Sinai (1993))
$\frac{q(t)}{t} \to J(E)$ where $J(E) = \frac{1}{2} D^2 E + o(|E|)$

Field w/o theremostate: (Chernov-D. (2009)) $\frac{\langle q(t), E\rangle^{3/2}}{t} \Rightarrow \Gamma\left(\frac{2}{3}\right)$

Infinite horizon:

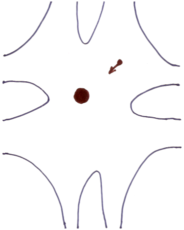
Free motion: (Szasz-Varju (2007)) $\frac{q(t)}{\sqrt{t \ln t}} \Rightarrow \mathcal{N}(0, \mathbb{D}^2)$.

Field with thermostate: (Chernov-Dolgopyat (2009))
$\frac{q(t)}{t} \to J(E)$ where $J(E) = \frac{|\ln E|}{2} \mathbb{D}^2 E + O(|E|)$.

Field without theremostate: ???

Two particle model of Brownian Motion

Consider 2 particles P and p moving in a dispersing domain $\mathcal{D}$ in $\mathbb{T}^2$ and collidong elastically with the walls and with each others.



	size	mass	init. velocity	init. position
P	r	$M \gg 1$	0	fixed
p	0	1	1	random

Theorem (Chernov-Dolgopyat (2009))
(a) As $M \to \infty$ $(Q(\tau M^{2/3}), M^{2/3}V(\tau M^{2/3}) \Rightarrow (\mathcal{Q}, \mathcal{V})(\tau)$ where

$$d\mathcal{Q} = \mathcal{V} d\tau, \quad d\mathcal{V} = \sigma(\mathcal{Q}) dW(\tau).$$

(b) As $r \to 0$ we have $\sigma^2(\mathcal{Q}) = \frac{8r}{3\text{Area}(\mathcal{D})} + o(r)$.

Conclusion.

- Kolya played a major role in converting the theory of chaotic billiards from a new mathematical field to the state where realisitic physical models can be analyzed mathematically.
- Kolya's contributions go far beyond billiards. Many standard tools in dynamics go back to his papers.
- Several directions started by Kolya remain active areas of research. Selected open problems are
  - Statistical properties of hard ball systems
  - Effective equations for many-particle systems
  - Mixing rates for hyperbolic flows and other systems with symmetries ...

Conclusion.

- Kolya played a major role in converting the theory of chaotic billiards from a new mathematical field to the state where realisitic physical models can be analyzed mathematically.
- Kolya's contributions go far beyond billiards. Many standard tools in dynamics go back to his papers.
- Several directions started by Kolya remain active areas of research. Selected open problems are
  - Statistical properties of hard ball systems
  - Effective equations for many-particle systems
  - Mixing rates for hyperbolic flows and other systems with symmetries ...

Kolya's departure is a great loss to the mathematical community.