DYNAMICAL BOREL-CANTELLI LEMMAS

Dmitry Kleinbock and Nikolai Chernov

1999–2015

Motivation 1: Borel-Cantelli Lemma

Motivation 1: Borel-Cantelli Lemma

Given a probability space (X, μ), a sequence of subsets A_k of X and $x \in X$, look at the number of sets A_k that contain x:

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid x \in A_k\} = \sum_{k=1}^{\infty} 1_{A_k}(x)$$

Motivation 1: Borel-Cantelli Lemma

Given a probability space (X, μ), a sequence of subsets A_k of X and $x \in X$, look at the number of sets A_k that contain x:

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid x \in A_k\} = \sum_{k=1}^{\infty} 1_{A_k}(x)$$

(i) If $\sum \mu(A_k) < \infty$, then $S_\infty(x) \underset{\text{a.e.}}{<} \infty$, i.e. almost every point $x \in X$ belongs to finitely many A_k.

Motivation 1: Borel-Cantelli Lemma

Given a probability space (X, μ), a sequence of subsets A_k of X and $x \in X$, look at the number of sets A_k that contain x:

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid x \in A_k\} = \sum_{k=1}^{\infty} 1_{A_k}(x)$$

(i) If $\sum \mu(A_k) < \infty$, then $S_\infty(x) \underset{\text{a.e.}}{<} \infty$, i.e. almost every point $x \in X$ belongs to finitely many A_k.

(ii) If $\sum \mu(A_k) = \infty$ and $\underline{A_k \text{ are independent}}$, then $S_\infty(x) \underset{\text{a.e.}}{=} \infty$,

Motivation 1: Borel-Cantelli Lemma

Given a probability space (X, μ), a sequence of subsets A_k of X and $x \in X$, look at the number of sets A_k that contain x:

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid x \in A_k\} = \sum_{k=1}^{\infty} 1_{A_k}(x)$$

(i) If $\sum \mu(A_k) < \infty$, then $S_\infty(x) \underset{\text{a.e.}}{<} \infty$, i.e. almost every point $x \in X$ belongs to finitely many A_k.

(ii) If $\sum \mu(A_k) = \infty$ and $\underline{A_k \text{ are independent}}$, then $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, i.e. almost every point $x \in X$ belongs to infinitely many A_k. Furthermore,

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\to} 1 \text{ as } N \to \infty\,,$$

Motivation 1: Borel-Cantelli Lemma

Given a probability space (X, μ), a sequence of subsets A_k of X and $x \in X$, look at the number of sets A_k that contain x:

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid x \in A_k\} = \sum_{k=1}^{\infty} 1_{A_k}(x)$$

(i) If $\sum \mu(A_k) < \infty$, then $S_\infty(x) \underset{\text{a.e.}}{<} \infty$, i.e. almost every point $x \in X$ belongs to finitely many A_k.

(ii) If $\sum \mu(A_k) = \infty$ and $\underline{A_k \text{ are independent}}$, then $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, i.e. almost every point $x \in X$ belongs to infinitely many A_k.
Furthermore,

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\to} 1 \text{ as } N \to \infty\,,$$

Here $S_N(x) \stackrel{\text{def}}{=} \#\{1 \le k \le N \mid x \in A_k\} = \sum_{k=1}^{N} 1_{A_k}(x)$
and $E_N \stackrel{\text{def}}{=} \sum_{k=1}^{N} \mu(A_k) = E[S_N]$.

Motivation 2: Birkhoff's Theorem

Motivation 2: Birkhoff's Theorem

$$T : (X, \mu) \circlearrowleft \text{ ergodic}$$

Motivation 2: Birkhoff's Theorem

$$T : (X, \mu) \circlearrowleft \text{ ergodic}$$

$$\Updownarrow$$

$$\forall\, B \subset X \text{ with } \mu(B) > 0, \text{ define } A_k \stackrel{\text{def}}{=} T^{-k}(B); \text{ then}$$

Motivation 2: Birkhoff's Theorem

$$T : (X, \mu) \circlearrowleft \text{ ergodic}$$

$$\Updownarrow$$

$\forall\, B \subset X$ with $\mu(B) > 0$, define $A_k \stackrel{\text{def}}{=} T^{-k}(B)$; then

$$\frac{S_N(x)}{E_N} = \frac{\#\{1 \le k \le N \mid T^k x \in B\}}{N\mu(B)} \underset{\text{a.e.}}{\to} 1 \text{ as } N \to \infty$$

Motivation 2: Birkhoff's Theorem

$$T : (X, \mu) \circlearrowleft \text{ ergodic}$$

$$\Updownarrow$$

$\forall\, B \subset X$ with $\mu(B) > 0$, define $A_k \stackrel{\text{def}}{=} T^{-k}(B)$; then

$$\frac{S_N(x)}{E_N} = \frac{\#\{1 \le k \le N \mid T^k x \in B\}}{N\mu(B)} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty$$

$$\Updownarrow$$

$$S_\infty(x) = \#\{k \in \mathbb{N} \mid T^k x \in B\} \underset{\text{a.e.}}{=} \infty$$

Motivation 3: an early shrinking target theorem

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$,

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$, or
- $T(x) = \{1/x\}$ (the Gauss transformation),

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$, or
- $T(x) = \{1/x\}$ (the Gauss transformation),

and let μ be the unique T-invariant smooth measure on $[0,1]$.

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$, or
- $T(x) = \{1/x\}$ (the Gauss transformation),

and let μ be the unique T-invariant smooth measure on $[0,1]$.

Take any sequence of subintervals $\{B_k\}$ of $[0,1]$ with $\sum \mu(B_k) = \infty$, and let $A_k \stackrel{\text{def}}{=} T^{-k}(B_k)$.

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$, or
- $T(x) = \{1/x\}$ (the Gauss transformation),

and let μ be the unique T-invariant smooth measure on $[0,1]$.

Take any sequence of subintervals $\{B_k\}$ of $[0,1]$ with $\sum \mu(B_k) = \infty$, and let $A_k \stackrel{\text{def}}{=} T^{-k}(B_k)$. Then

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty.$$

Motivation 3: an early shrinking target theorem

[Philipp 1969]: Take $T : [0,1] \circlearrowleft$ given by

- $T(x) = \beta x \pmod 1$ with $\beta > 1$, or
- $T(x) = \{1/x\}$ (the Gauss transformation),

and let μ be the unique T-invariant smooth measure on $[0,1]$.

Take any sequence of subintervals $\{B_k\}$ of $[0,1]$ with $\sum \mu(B_k) = \infty$, and let $A_k \stackrel{\text{def}}{=} T^{-k}(B_k)$. Then

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty.$$

This, in particular, gives the optimal rate of approximation of arbitrary point of $[0,1]$ by orbit points $T^k x$ for a.e. $x \in [0,1]$.

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Let m, n be positive integers and $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing.

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Let m, n be positive integers and $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing. Define

$$A_k = \left\{ Y \in [0,1]^{m\times n} \;\middle|\; \begin{array}{l} \|Y\mathbf{q} + \mathbf{p}\| \le \psi(\|\mathbf{q}\|) \\ \text{for some } \mathbf{p} \in \mathbb{Z}^m \text{ and} \\ \mathbf{q} \in \mathbb{Z}^n \text{ with } \|\mathbf{q}\| = k \end{array} \right\},$$

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Let m, n be positive integers and $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing. Define

$$A_k = \left\{ Y \in [0,1]^{m\times n} \;\middle|\; \begin{array}{l} \|Y\mathbf{q}+\mathbf{p}\| \le \psi(\|\mathbf{q}\|) \\ \text{for some } \mathbf{p} \in \mathbb{Z}^m \text{ and} \\ \mathbf{q} \in \mathbb{Z}^n \text{ with } \|\mathbf{q}\| = k \end{array} \right\},$$

and assume that

$$\sum_{k=1}^{\infty} k^{m-1}\psi^n(k) \asymp E_\infty = \infty.$$

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Let m, n be positive integers and $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing. Define

$$A_k = \left\{ Y \in [0,1]^{m \times n} \,\middle|\, \begin{array}{l} \|Y\mathbf{q} + \mathbf{p}\| \le \psi(\|\mathbf{q}\|) \\ \text{for some } \mathbf{p} \in \mathbb{Z}^m \text{ and} \\ \mathbf{q} \in \mathbb{Z}^n \text{ with } \|\mathbf{q}\| = k \end{array} \right\},$$

and assume that

$$\sum_{k=1}^{\infty} k^{m-1} \psi^n(k) \asymp E_\infty = \infty.$$

Then

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty.$$

Motivation 4: the Khintchine-Groshev-Schmidt Theorem

Let m, n be positive integers and $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing. Define

$$A_k = \left\{ Y \in [0,1]^{m\times n} \;\middle|\; \begin{array}{l} \|Y\mathbf{q} + \mathbf{p}\| \leq \psi(\|\mathbf{q}\|) \\ \text{for some } \mathbf{p} \in \mathbb{Z}^m \text{ and} \\ \mathbf{q} \in \mathbb{Z}^n \text{ with } \|\mathbf{q}\| = k \end{array} \right\},$$

and assume that

$$\sum_{k=1}^{\infty} k^{m-1}\psi^n(k) \asymp E_\infty = \infty.$$

Then

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty.$$

In particular, almost every Y lies in infinitely many A_k.

Even though in the above theorem the sets A_k are not in the form $T^{-k}B_k$, in [K–Margulis 1999] it was explained, following an earlier work of [Sullivan 1982] and [Dani 1985], how the set-up of the previous slide is related to certain flows on the homogeneous space of unimodular lattices in $\mathbb{R}^{m+n}$.

Even though in the above theorem the sets A_k are not in the form $T^{-k}B_k$, in [K–Margulis 1999] it was explained, following an earlier work of [Sullivan 1982] and [Dani 1985], how the set-up of the previous slide is related to certain flows on the homogeneous space of unimodular lattices in $\mathbb{R}^{m+n}$.

Following that work, we met with Kolya and started thinking about what else could be done...

Definition

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving.

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

if for μ-a.e. $x \in X$ there are infinitely many k such that $T^k x \in B_k$;

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

if for μ-a.e. $x \in X$ there are infinitely many k such that $T^k x \in B_k$; in other words, if $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, where

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid T^k x \in B_k\} = \sum_{n=1}^{\infty} 1_{A_k}(x), \; A_k = T^{-k} B_k.$$

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

if for μ-a.e. $x \in X$ there are infinitely many k such that $T^k x \in B_k$; in other words, if $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, where

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid T^k x \in B_k\} = \sum_{n=1}^{\infty} 1_{A_k}(x), \;\; A_k = T^{-k} B_k.$$

Also say that $\{B_k\}$ is a *strongly Borel-Cantelli* (sBC) sequence (relative to T)

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

if for μ-a.e. $x \in X$ there are infinitely many k such that $T^k x \in B_k$; in other words, if $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, where

$$S_\infty(x) \stackrel{\text{def}}{=} \#\{k \in \mathbb{N} \mid T^k x \in B_k\} = \sum_{n=1}^{\infty} 1_{A_k}(x), \; A_k = T^{-k} B_k.$$

Also say that $\{B_k\}$ is a *strongly Borel-Cantelli* (sBC) sequence (relative to T) if

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\longrightarrow} 1 \text{ as } N \to \infty$$

where S_N and E_N are defined as before,

Definition

Let $T : (X, \mu) \circlearrowleft$ be measure preserving. Say that a sequence of subsets B_k of X is a

Borel-Cantelli (BC) sequence (relative to T)

if for μ-a.e. $x \in X$ there are infinitely many k such that $T^k x \in B_k$; in other words, if $S_\infty(x) \underset{\text{a.e.}}{=} \infty$, where

$$S_\infty(x) \overset{\text{def}}{=} \#\{k \in \mathbb{N} \mid T^k x \in B_k\} = \sum_{n=1}^{\infty} 1_{A_k}(x), \; A_k = T^{-k} B_k.$$

Also say that $\{B_k\}$ is a *strongly Borel-Cantelli* (sBC) sequence (relative to T) if

$$\frac{S_N(x)}{E_N} \underset{\text{a.e.}}{\to} 1 \text{ as } N \to \infty$$

where S_N and E_N are defined as before,

(A necessary condition: $E_\infty = \infty$, will always assume that.)

What is this notion good for?

Ergodicity Criterion: by Birkhoff's Theorem, T is ergodic

What is this notion good for?

Ergodicity Criterion: by Birkhoff's Theorem, T is ergodic

$$\Updownarrow$$

every constant sequence $B_k \equiv B$, $\mu(B) > 0$, is BC

What is this notion good for?

Ergodicity Criterion: by Birkhoff's Theorem, T is ergodic

$$\Updownarrow$$

every constant sequence $B_k \equiv B$, $\mu(B) > 0$, is BC

$$\Updownarrow$$

every such sequence is sBC.

What is this notion good for?

Ergodicity Criterion: by Birkhoff's Theorem, T is ergodic

$$\Updownarrow$$

every constant sequence $B_k \equiv B$, $\mu(B) > 0$, is BC

$$\Updownarrow$$

every such sequence is sBC.

Weak Mixing Criterion [Chernov-K 2001]: T is weakly mixing

What is this notion good for?

Ergodicity Criterion: by Birkhoff's Theorem, T is ergodic

$$\Updownarrow$$

every constant sequence $B_k \equiv B$, $\mu(B) > 0$, is BC

$$\Updownarrow$$

every such sequence is sBC.

Weak Mixing Criterion [Chernov-K 2001]: T is weakly mixing

$$\Updownarrow$$

every sequence $\{B_k\}$ that contains only finitely many distinct sets, none of them of measure zero, is BC.

Proof. Choose $\{B_k\}$ from $F_1, \ldots, F_\ell$;

Proof. Choose $\{B_k\}$ from $F_1, \ldots, F_\ell$; by weak mixing, $\forall\, i, j$

$$\sum_{k=1}^{N} |\mu(T^{-k}F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)$$

Proof. Choose $\{B_k\}$ from $F_1, \ldots, F_\ell$; by weak mixing, $\forall\, i, j$

$$\sum_{k=1}^{N} |\mu(T^{-k} F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)$$

$$\Downarrow$$

$$E\left[(S_N - E_N)^2\right] \leq 2 \sum_{k=1}^{N} \sum_{\ell=k}^{N} \Big(\mu(T^{-(\ell-k)} B_\ell \cap B_k) - \mu(B_s)\mu(B_r) \Big)$$
$$= o(N^2) \qquad \text{and } E_N \asymp N$$

Proof. Choose $\{B_k\}$ from $F_1, \dots, F_\ell$; by weak mixing, $\forall\, i, j$

$$\sum_{k=1}^{N} |\mu(T^{-k} F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)$$

$$\Downarrow$$

$$E\left[(S_N - E_N)^2\right] \leq 2 \sum_{k=1}^{N} \sum_{\ell=k}^{N} \Big(\mu(T^{-(\ell-k)} B_\ell \cap B_k) - \mu(B_s)\mu(B_r) \Big)$$
$$= o(N^2) \qquad \text{and } E_N \asymp N$$

$$\Downarrow$$

$$E\left[\left(\frac{S_N}{E_N} - 1\right)^2\right] \to 0 \quad \text{as} \quad N \to \infty$$

Proof. Choose $\{B_k\}$ from $F_1, \ldots, F_\ell$; by weak mixing, $\forall\, i, j$

$$\sum_{k=1}^{N} |\mu(T^{-k} F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)$$

$$\Downarrow$$

$$E\left[(S_N - E_N)^2\right] \leq 2 \sum_{k=1}^{N} \sum_{\ell=k}^{N} \Big(\mu(T^{-(\ell-k)} B_\ell \,\cap\, B_k) - \mu(B_s)\mu(B_r)\Big)$$

$$= o(N^2) \qquad \text{and } E_N \asymp N$$

$$\Downarrow$$

$$E\left[\left(\frac{S_N}{E_N} - 1\right)^2\right] \to 0 \quad \text{as} \quad N \to \infty$$

$$\Downarrow$$

for some subsequence $\{N_k\}$, $S_{N_k}/E_{N_k} \underset{\text{a.e.}}{\longrightarrow} 1 \Rightarrow S_\infty \underset{\text{a.e.}}{=} \infty$.

Proof. Choose $\{B_k\}$ from $F_1, \ldots, F_\ell$; by weak mixing, $\forall\, i, j$

$$\sum_{k=1}^{N} |\mu(T^{-k} F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)$$

$$\Downarrow$$

$$E\left[(S_N - E_N)^2\right] \leq 2 \sum_{k=1}^{N} \sum_{\ell=k}^{N} \Big(\mu(T^{-(\ell-k)} B_\ell \cap B_k) - \mu(B_s)\mu(B_r) \Big)$$
$$= o(N^2) \qquad \text{and } E_N \asymp N$$

$$\Downarrow$$

$$E\left[\left(\frac{S_N}{E_N} - 1\right)^2\right] \to 0 \quad \text{as} \quad N \to \infty$$

$$\Downarrow$$

for some subsequence $\{N_k\}$, $S_{N_k}/E_{N_k} \underset{\text{a.e.}}{\longrightarrow} 1 \Rightarrow S_\infty \underset{\text{a.e.}}{=} \infty$.

Converse – by looking at irrational rotations. □

There are always non-BC sequences:

There are always non-BC sequences:

Proposition [Chernov-K 2001]. If μ is non-atomic, then for any μ-preserving transformation T of X there exists a sequence $\{B_k\}$ with $E_\infty = \infty$ and $S_\infty \underset{\text{a.e.}}{<} \infty$, hence not BC.

There are always non-BC sequences:

Proposition [Chernov-K 2001]. If μ is non-atomic, then for any μ-preserving transformation T of X there exists a sequence $\{B_k\}$ with $E_\infty = \infty$ and $S_\infty \underset{\text{a.e.}}{<} \infty$, hence not BC.

Proof. Start with $\{A_n\}$ with convergent some of measures, then derive. □

There are always non-BC sequences:

Proposition [Chernov-K 2001]. If μ is non-atomic, then for any μ-preserving transformation T of X there exists a sequence $\{B_k\}$ with $E_\infty = \infty$ and $S_\infty \underset{\text{a.e.}}{<} \infty$, hence <u>not</u> BC.

Proof. Start with $\{A_n\}$ with convergent some of measures, then derive. □

Therefore to prove BC or sBC properties for certain classes of sequences (containing infinitely many distinct sets) it is necessary to impose certain restrictions on the sets B_k.

A natural restriction:

A natural restriction:

for a metric space X (e.g. a Riemannian manifold)

A natural restriction:

for a metric space X (e.g. a Riemannian manifold) and some $T : X \circlearrowleft$,

A natural restriction:

for a metric space X (e.g. a Riemannian manifold) and some $T : X \circlearrowleft$, one can try to prove that

all sequences of balls in X are BC or sBC

A natural restriction:

for a metric space X (e.g. a Riemannian manifold)
and some $T : X \circlearrowleft$, one can try to prove that

all sequences of balls in X are BC or sBC

If this is the case, one can take any $x_0 \in X$ and consider what could be called

"a target shrinking to x_0"

A natural restriction:

for a metric space X (e.g. a Riemannian manifold) and some $T : X \circlearrowleft$, one can try to prove that

all sequences of balls in X are BC or sBC

If this is the case, one can take any $x_0 \in X$ and consider what could be called

"a target shrinking to x_0"

i.e. a sequence of balls $B_k = B(x_0, r_k)$ with $r_k \to 0$.

A natural restriction:

for a metric space X (e.g. a Riemannian manifold) and some $T : X \circlearrowleft$, one can try to prove that

all sequences of balls in X are BC or sBC

If this is the case, one can take any $x_0 \in X$ and consider what could be called

"a target shrinking to x_0"

i.e. a sequence of balls $B_k = B(x_0, r_k)$ with $r_k \to 0$.

Then almost all orbits $\{T^k x\}$ will get into infinitely many such balls whenever r_k decays slowly enough $\Rightarrow$ a quantitative strengthening of density of almost all orbits (in other words, all points $x_0 \in X$ can be "well approximated" by orbit points $T^k x$ for almost all x).

A natural restriction:

for a metric space X (e.g. a Riemannian manifold)
and some $T : X \circlearrowleft$, one can try to prove that

all sequences of balls in X are BC or sBC

If this is the case, one can take any $x_0 \in X$ and consider what could be called

"a target shrinking to x_0"

i.e. a sequence of balls $B_k = B(x_0, r_k)$ with $r_k \to 0$.

Then almost all orbits $\{T^k x\}$ will get into infinitely many such balls whenever r_k decays slowly enough $\Rightarrow$ a quantitative strengthening of density of almost all orbits (in other words, all points $x_0 \in X$ can be "well approximated" by orbit points $T^k x$ for almost all x).

(**Variants**: sequences of neighborhoods of other sets;
under an additional assumption of E_N diverging fast enough).

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k - \ell|$.

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindǎuk] Assume that

$$\exists C > 0 : \quad E\left[(S_{M,N} - E_{M,N})^2\right] \le C \cdot E_{M,N} \text{ for all } N \ge M \ge 1. \quad (1)$$

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindžuk] Assume that

$$\exists C > 0 : \quad E\left[(S_{M,N} - E_{M,N})^2\right] \le C \cdot E_{M,N} \text{ for all } N \ge M \ge 1. \quad (1)$$

(Here, as before, $A_k = T^{-k}B_k$, and also

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindǎuk] Assume that

$$\exists C > 0 : \quad E\left[(S_{M,N} - E_{M,N})^2\right] \leq C \cdot E_{M,N} \text{ for all } N \geq M \geq 1. \quad (1)$$

(Here, as before, $A_k = T^{-k}B_k$, and also

$S_{M,N}(x) \stackrel{\text{def}}{=} \#\{M+1 \leq k \leq N \mid x \in A_k\} = \sum_{k=M+1}^{N} 1_{A_k}(x)$,

$E_{M,N} \stackrel{\text{def}}{=} \sum_{k=M+1}^{N} \mu(A_k) = E[S_{M,N}]$.)

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindǎuk] Assume that

$$\exists C > 0 : \quad E\left[(S_{M,N} - E_{M,N})^2\right] \le C \cdot E_{M,N} \text{ for all } N \ge M \ge 1. \quad (1)$$

(Here, as before, $A_k = T^{-k}B_k$, and also

$S_{M,N}(x) \stackrel{\text{def}}{=} \#\{M+1 \le k \le N \mid x \in A_k\} = \sum_{k=M+1}^{N} 1_{A_k}(x)$,

$E_{M,N} \stackrel{\text{def}}{=} \sum_{k=M+1}^{N} \mu(A_k) = E[S_{M,N}]$.)

Then $\forall\, \varepsilon > 0$

$$S_N \underset{\text{a.e.}}{=} E_N + O\left(E_N^{1/2} \log^{3/2+\varepsilon} E_N\right). \quad (2)$$

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindǎuk] Assume that

$$\exists C > 0 : \quad E\left[(S_{M,N} - E_{M,N})^2\right] \le C \cdot E_{M,N} \text{ for all } N \ge M \ge 1. \qquad (1)$$

(Here, as before, $A_k = T^{-k}B_k$, and also

$S_{M,N}(x) \stackrel{\text{def}}{=} \#\{M+1 \le k \le N \mid x \in A_k\} = \sum_{k=M+1}^{N} 1_{A_k}(x)$,

$E_{M,N} \stackrel{\text{def}}{=} \sum_{k=M+1}^{N} \mu(A_k) = E[S_{M,N}]$.)

Then $\forall\, \varepsilon > 0$

$$S_N \underset{\text{a.e.}}{=} E_N + O\left(E_N^{1/2} \log^{3/2+\varepsilon} E_N\right). \qquad (2)$$

In particular (assuming $E_\infty = \infty$) $\{B_k\}$ is an sBC sequence.

Tool: quasi-independence of translates $T^{-k}B_k$ and $T^{-\ell}B_\ell$ for large $|k-\ell|$.

Lemma [?–Cassels–Schmidt–Sprindǎuk] Assume that

$$\exists C>0: \quad E\left[(S_{M,N}-E_{M,N})^2\right] \le C\cdot E_{M,N} \text{ for all } N\ge M\ge 1. \qquad (1)$$

(Here, as before, $A_k = T^{-k}B_k$, and also

$S_{M,N}(x) \stackrel{\text{def}}{=} \#\{M+1\le k\le N \mid x\in A_k\} = \sum_{k=M+1}^{N} 1_{A_k}(x)$,

$E_{M,N} \stackrel{\text{def}}{=} \sum_{k=M+1}^{N} \mu(A_k) = E[S_{M,N}]$.)

Then $\forall\, \varepsilon>0$

$$S_N \underset{\text{a.e.}}{=} E_N + O\left(E_N^{1/2}\log^{3/2+\varepsilon} E_N\right). \qquad (2)$$

In particular (assuming $E_\infty=\infty$) $\{B_k\}$ is an sBC sequence.

Proof. Chebyshev's Inequality and a carefully arranged subdivision of $\{1,\dots,N\}$. □

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed,

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N} - E_{M,N})^2\right] = \sum_{k,\ell=M+1}^{N} E\left[\left(1_{A_k} - \mu(A_k)\right)\left(1_{A_\ell} - \mu(A_\ell)\right)\right]$$

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N}-E_{M,N})^2\right]=\sum_{k,\ell=M+1}^{N}E\left[\left(1_{A_k}-\mu(A_k)\right)\left(1_{A_\ell}-\mu(A_\ell)\right)\right]$$

$$\leq 2\sum_{\ell=M+1}^{N}\sum_{k=\ell}^{N}\Big(\mu\big(T^{-(k-\ell)}B_k\cap B_\ell\big)-\mu(B_k)\mu(B_\ell).\Big)$$

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N}-E_{M,N})^2\right]=\sum_{k,\ell=M+1}^{N} E\left[\left(1_{A_k}-\mu(A_k)\right)\left(1_{A_\ell}-\mu(A_\ell)\right)\right]$$

$$\leq 2\sum_{\ell=M+1}^{N}\sum_{k=\ell}^{N}\left(\mu\left(T^{-(k-\ell)}B_k\cap B_\ell\right)-\mu(B_k)\mu(B_\ell).\right)$$

Hence if the system is mixing with fast enough rate, there is a hope of verifying its shrinking target properties.

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N}-E_{M,N})^2\right]=\sum_{k,\ell=M+1}^{N}E\left[\left(1_{A_k}-\mu(A_k)\right)\left(1_{A_\ell}-\mu(A_\ell)\right)\right]$$

$$\leq 2\sum_{\ell=M+1}^{N}\sum_{k=\ell}^{N}\Big(\mu\big(T^{-(k-\ell)}B_k\cap B_\ell\big)-\mu(B_k)\mu(B_\ell).\Big)$$

Hence if the system is mixing with fast enough rate, there is a hope of verifying its shrinking target properties. This was the logic behind the papers of Philipp, Sullivan, K-Margulis.

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N}-E_{M,N})^2\right]=\sum_{k,\ell=M+1}^{N}E\left[\left(1_{A_k}-\mu(A_k)\right)\left(1_{A_\ell}-\mu(A_\ell)\right)\right]$$

$$\leq 2\sum_{\ell=M+1}^{N}\sum_{k=\ell}^{N}\Big(\mu\big(T^{-(k-\ell)}B_k\cap B_\ell\big)-\mu(B_k)\mu(B_\ell).\Big)$$

Hence if the system is mixing with fast enough rate, there is a hope of verifying its shrinking target properties. This was the logic behind the papers of Philipp, Sullivan, K-Margulis.

So when Kolya visited Rutgers in 1999, I approached him, as an expert on decay of correlations, and suggested to explore this theme together for some other dynamical systems.

The inequality in the above lemma can often be derived from the decay of correlations of the dynamical system. Indeed, one has

$$E\left[(S_{M,N} - E_{M,N})^2\right] = \sum_{k,\ell=M+1}^{N} E\left[\left(1_{A_k} - \mu(A_k)\right)\left(1_{A_\ell} - \mu(A_\ell)\right)\right]$$

$$\leq 2\sum_{\ell=M+1}^{N}\sum_{k=\ell}^{N}\Big(\mu\big(T^{-(k-\ell)}B_k \cap B_\ell\big) - \mu(B_k)\mu(B_\ell).\Big)$$

Hence if the system is mixing with fast enough rate, there is a hope of verifying its shrinking target properties. This was the logic behind the papers of Philipp, Sullivan, K-Margulis.

So when Kolya visited Rutgers in 1999, I approached him, as an expert on decay of correlations, and suggested to explore this theme together for some other dynamical systems.

For example, starting with symbolic dynamics...

Sequences and cylinders

Sequences and cylinders

Let $\mathbf{A}$ be a transitive stochastic matrix and let $\Sigma = \Sigma_{\mathbf{A}}$ be the topological Markov chain given by $\mathbf{A}$:

$$\Sigma = \{\underline{\omega} \in \{1, \ldots, M\}^{\mathbb{Z}} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}, \ \sigma := \text{the left shift.}$$

Sequences and cylinders

Let $\mathbf{A}$ be a transitive stochastic matrix and let $\Sigma = \Sigma_{\mathbf{A}}$ be the topological Markov chain given by $\mathbf{A}$:

$$\Sigma = \{\underline{\omega} \in \{1, \ldots, M\}^{\mathbb{Z}} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}, \ \sigma := \text{the left shift.}$$

It is a compact metric space, with distance

$$d(\underline{\omega}, \underline{\omega}') = \left(\frac{1}{2}\right)^{\max\{n:\, \omega_i = \omega_i',\ \forall |i| < n\}} .$$

Sequences and cylinders

Let $\mathbf{A}$ be a transitive stochastic matrix and let $\Sigma = \Sigma_{\mathbf{A}}$ be the topological Markov chain given by $\mathbf{A}$:

$$\Sigma = \{\underline{\omega} \in \{1, \ldots, M\}^{\mathbb{Z}} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}, \ \sigma := \text{the left shift.}$$

It is a compact metric space, with distance

$$d(\underline{\omega}, \underline{\omega}') = \left(\frac{1}{2}\right)^{\max\{n:\, \omega_i = \omega_i',\ \forall |i| < n\}} .$$

A cylinder $C(\omega_\Lambda) \subset \Sigma$ is obtained by fixing symbols of $\underline{\omega} \in \Sigma$ on a finite interval $\Lambda = [n^-, n^+] \subset \mathbb{Z}$,

Sequences and cylinders

Let $\mathbf{A}$ be a transitive stochastic matrix and let $\Sigma = \Sigma_{\mathbf{A}}$ be the topological Markov chain given by $\mathbf{A}$:

$$\Sigma = \{\underline{\omega} \in \{1, \dots, M\}^{\mathbb{Z}} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}, \ \sigma := \text{the left shift.}$$

It is a compact metric space, with distance

$$d(\underline{\omega}, \underline{\omega}') = \left(\frac{1}{2}\right)^{\max\{n:\, \omega_i = \omega_i',\ \forall |i| < n\}} .$$

A cylinder $C(\omega_\Lambda) \subset \Sigma$ is obtained by fixing symbols of $\underline{\omega} \in \Sigma$ on a finite interval $\Lambda = [n^-, n^+] \subset \mathbb{Z}$,

i.e. for $\omega_\Lambda = \{\omega_{n^-}, \dots, \omega_{n^+}\} \in \{1, \dots, M\}^\Lambda$ we set

$$C(\omega_\Lambda) := \{\underline{\omega}' \in \Sigma : \ \omega_i' = \omega_i \quad \text{for} \ \ n^- \le i \le n^+\}$$

Gibbs measures

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\left(C(\omega_{[1,N]})\right)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\left(\sigma^k(\underline{\omega})\right)\right)} \leq a_2.$$

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\right)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\right)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Main Theorem [Chernov-K 2001] Let $\{B_k\}$ be a sequence of cylinders defined on intervals $\Lambda_k \subset \mathbb{Z}$.

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\right)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Main Theorem [Chernov-K 2001] Let $\{B_k\}$ be a sequence of cylinders defined on intervals $\Lambda_k \subset \mathbb{Z}$. Let $D \geq 0$ be a constant, and let μ be a Gibbs measure.

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\right)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Main Theorem [Chernov-K 2001] Let $\{B_k\}$ be a sequence of cylinders defined on intervals $\Lambda_k \subset \mathbb{Z}$. Let $D \geq 0$ be a constant, and let μ be a Gibbs measure. Assume that for all k, ℓ the intervals Λ_k, Λ_ℓ are D-nested.

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\left(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\right)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Main Theorem [Chernov-K 2001] Let $\{B_k\}$ be a sequence of cylinders defined on intervals $\Lambda_k \subset \mathbb{Z}$. Let $D \geq 0$ be a constant, and let μ be a Gibbs measure. Assume that for all k, ℓ the intervals Λ_k, Λ_ℓ are D-nested. Then the sequence $\{B_k\}$ satisfies (1)

Gibbs measures

Theorem=Definition. [Bowen] For any Hölder continuous potential $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ on Σ and constants $a_1, a_2 > 0$ and P (the topological pressure of ψ) such that for every $\underline{\omega} \in \Sigma$ and $N \in \mathbb{N}$,

$$a_1 \leq \frac{\mu\big(C(\omega_{[1,N]})\big)}{\exp\Big(-PN + \sum_{k=1}^{N} \psi\big(\sigma^k(\underline{\omega})\big)\Big)} \leq a_2.$$

Definition. We say that two intervals Λ_1 and Λ_2 are *D-nested* for $D \geq 0$ if one is in D-neighborhood of the other.

Main Theorem [Chernov-K 2001] Let $\{B_k\}$ be a sequence of cylinders defined on intervals $\Lambda_k \subset \mathbb{Z}$. Let $D \geq 0$ be a constant, and let μ be a Gibbs measure. Assume that for all k, ℓ the intervals Λ_k, Λ_ℓ are D-nested. Then the sequence $\{B_k\}$ satisfies (1) and hence, if in addition $E_\infty = \infty$, it is an sBC sequence and (2) holds.

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \leq c\theta^L \mu(B_1)\mu(B_2)\,, \qquad (3)$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \le c\theta^L \mu(B_1)\mu(B_2)\,, \qquad (3)$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Proof of the Main Theorem.

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \leq c\theta^L \mu(B_1)\mu(B_2)\,, \tag{3}$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Proof of the Main Theorem. If B_1, B_2 are cylinders defined on D-nested intervals Λ_1 and Λ_2 respectively, then the gap between intervals defining cylinders $T^{-k}(B_1)$ and $T^{-\ell}(B_2)$ is at least $|\ell - k| - D$

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \le c\theta^L \mu(B_1)\mu(B_2)\,, \tag{3}$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Proof of the Main Theorem. If B_1, B_2 are cylinders defined on D-nested intervals Λ_1 and Λ_2 respectively, then the gap between intervals defining cylinders $T^{-k}(B_1)$ and $T^{-\ell}(B_2)$ is at least $|\ell - k| - D \Rightarrow$ the convergence in (1) follows from (3) and the D-nested assumption. □

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \le c\theta^L \mu(B_1)\mu(B_2)\,, \tag{3}$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Proof of the Main Theorem. If B_1, B_2 are cylinders defined on D-nested intervals Λ_1 and Λ_2 respectively, then the gap between intervals defining cylinders $T^{-k}(B_1)$ and $T^{-\ell}(B_2)$ is at least $|\ell - k| - D \Rightarrow$ the convergence in (1) follows from (3) and the D-nested assumption. □

Remark. The nesting assumption cannot be easily removed, there are examples of 'almost nested' non-BC sequences constructed in [Chernov-K 2001].

Fact. [Bowen] Let B_1, B_2 be cylinders defined on intervals in $\mathbb{Z}$ with gap at least L. Then

$$|\mu(B_1 \cap B_2) - \mu(B_1)\mu(B_2)| \leq c\theta^L \mu(B_1)\mu(B_2)\,, \tag{3}$$

where $c > 0$ and $0 < \theta < 1$ only depend on the Gibbs measure μ.

Proof of the Main Theorem. If B_1, B_2 are cylinders defined on D-nested intervals Λ_1 and Λ_2 respectively, then the gap between intervals defining cylinders $T^{-k}(B_1)$ and $T^{-\ell}(B_2)$ is at least $|\ell - k| - D \Rightarrow$ the convergence in (1) follows from (3) and the D-nested assumption. □

Remark. The nesting assumption cannot be easily removed, there are examples of 'almost nested' non-BC sequences constructed in [Chernov-K 2001].

Application: to Anosov diffeomorphisms via Markov partitions [Chernov-K 2001].

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)
- [Gupta–Nicol–Ott 2010, Haydn–Nicol–Persson–Vaienti 2011] non-uniformly hyperbolic dynamical systems

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)
- [Gupta–Nicol–Ott 2010, Haydn–Nicol–Persson–Vaienti 2011] non-uniformly hyperbolic dynamical systems
- [Chaika–Constantine 2012] rotations and interval exchanges (the first theorem of that kind is due to [Kurzweil 1955])

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)
- [Gupta–Nicol–Ott 2010, Haydn–Nicol–Persson–Vaienti 2011] non-uniformly hyperbolic dynamical systems
- [Chaika–Constantine 2012] rotations and interval exchanges (the first theorem of that kind is due to [Kurzweil 1955])
- [Dolgopyat–Fayad–Vinogradov 201?] total translations, using equidistribution on homogeneous spaces

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)
- [Gupta–Nicol–Ott 2010, Haydn–Nicol–Persson–Vaienti 2011] non-uniformly hyperbolic dynamical systems
- [Chaika–Constantine 2012] rotations and interval exchanges (the first theorem of that kind is due to [Kurzweil 1955])
- [Dolgopyat–Fayad–Vinogradov 201?] total translations, using equidistribution on homogeneous spaces
- and many many more...

The argument described above has been repeatedly exploited, with some modifications, in many subsequent papers:

- [Maucourant 2006] geodesic flows on hyperbolic manifolds
- [Kim 2007, Gouëzel 2007] non-uniformly expanding interval maps
- [Kim–Galatolo 2007] generic interval exchanges
- [Gorodnik–Shah 2010] flows on homogeneous spaces (generalizing Maucourant and deriving applications to number theory)
- [Gupta–Nicol–Ott 2010, Haydn–Nicol–Persson–Vaienti 2011] non-uniformly hyperbolic dynamical systems
- [Chaika–Constantine 2012] rotations and interval exchanges (the first theorem of that kind is due to [Kurzweil 1955])
- [Dolgopyat–Fayad–Vinogradov 201?] total translations, using equidistribution on homogeneous spaces
- and many many more...

THANK YOU!