Part 3: The Notorious Piston

Dynamics of a massive piston in an ideal gas

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Abstract. This survey is a study of a dynamical system consisting of a massive piston in a cubic container of large size L filled with an ideal gas. The piston has mass $M \sim L^2$ and undergoes elastic collisions with $N \sim L^3$ non-interacting gas particles of mass $m = 1$.

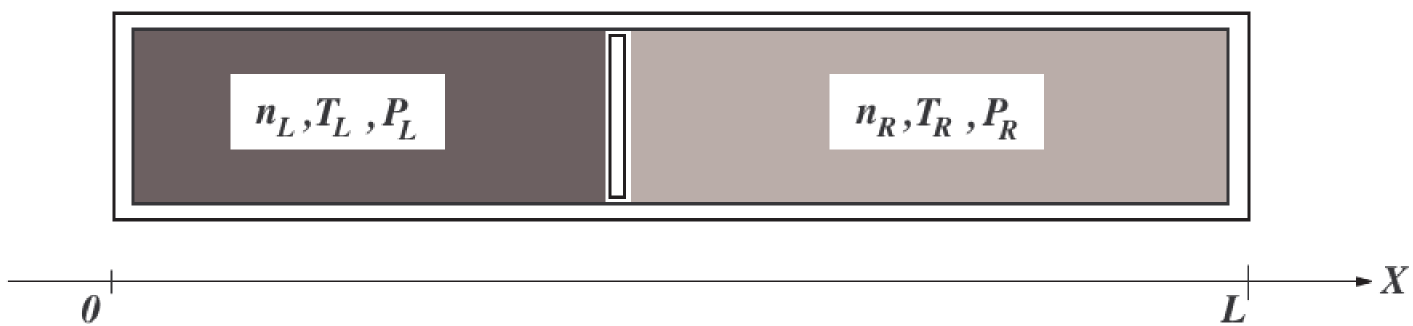

Figure 1: Piston in a cylinder filled with gas.

“ Starting with $P_L \neq P_R$, the piston moves under the net pressure difference and compresses the gas whose pressure is lower, until its pressure builds up and it pushes the piston back. Depending on the initial values of n_L, T_L, n_R, T_R and the dynamical characteristics of the gases, the piston may follow a complicated trajectory, sloshing back and forth, but gradually it comes to rest at a place where the pressures are equalized on both sides: $P_L = P_R$. At that time one also expects that the gas in each compartment will again be in equilibrium.

We observe, however, that the equality of pressures $P_L = P_R$ and the fact that the gas in each compartment separately is at equilibrium does not guarantee that $T_L = T_R$. In particular, for dilute gases (which we shall consider from now on), the pressure is related to the density and temperature by $P = nk_BT$, where k_B is Boltzmann's constant, and it is possible that $T_L < T_R$ while $n_L > n_R$, so that the gas in the left compartment is cooler but denser, and in the right one hotter but more dilute (or vice versa). ''

Why notorious ?

"Using the principle of maximum entropy Kubo [14], and Landau and Lifshitz [15], arrived at the conclusion that the condition for mechanical equilibrium is $\frac{p_1}{T_1} = \frac{p_2}{T_2}$, and they add that if heat transfer is also possible $T_1 = T_2$. Callen [3], on the other hand, arrives at the conclusion that the final pressures will be equal ($p_1 = p_2$), but the final temperatures T_1, T_2 will depend on the relative viscosity of each fluid and can not be predicted; Nozière [18] gives the same conclusion, but writes that the final temperatures will depend on the ratio of viscosities, . . . "

C. Gruber

Numerical Results

To clarify matters Kolya (with my kibitzing) carried out computer simulations on a system with equal density of particles on left and right with the piston in the middle and an initial distribution

$$p(x,v) = p(v) = \begin{cases} 1 & \text{if} \quad 0.5 \le |v| \le 1 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

so $v_{\text{min}} = 0.5$ and $v_{\text{max}} = 1$. The x and v coordinates of all the particles were independent random variables uniformly distributed in their ranges $0 < x < L$ and $v_{\text{min}} \le |v| \le v_{\text{max}}$. The parameter L varied in the simulations from $L = 30$ to $L = 300$. For $L = 300$ the system contains $\approx L^3 = 27,000,000$ particles.

Once the initial data is generated randomly, using a Poisson process the program computes the dynamics by using the elastic collision rules.

The figure below presents a typical trajectory of the piston. The position and time are measured in "hydrodynamic variables" $Y = X/L$, $0 < Y < 1$, and $\tau = t/L$.

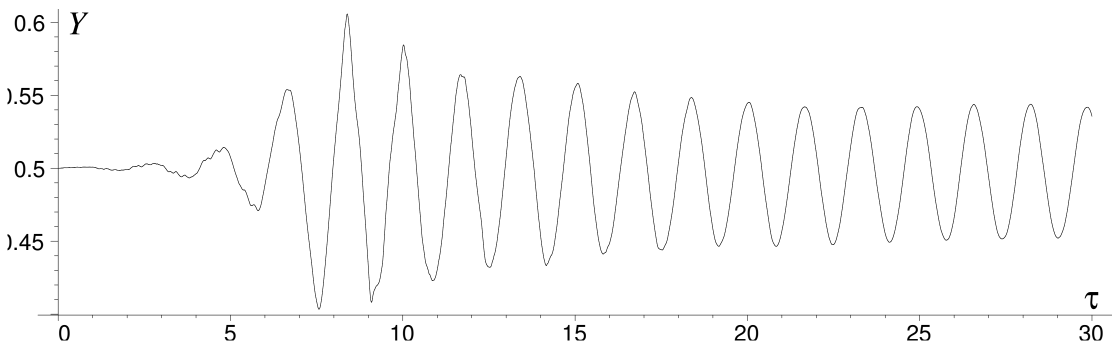

Figure 2: the piston coordinate Y as a function of time τ. Here $L = 100$, $N_- = 500341$, $N_+ = 499888$.

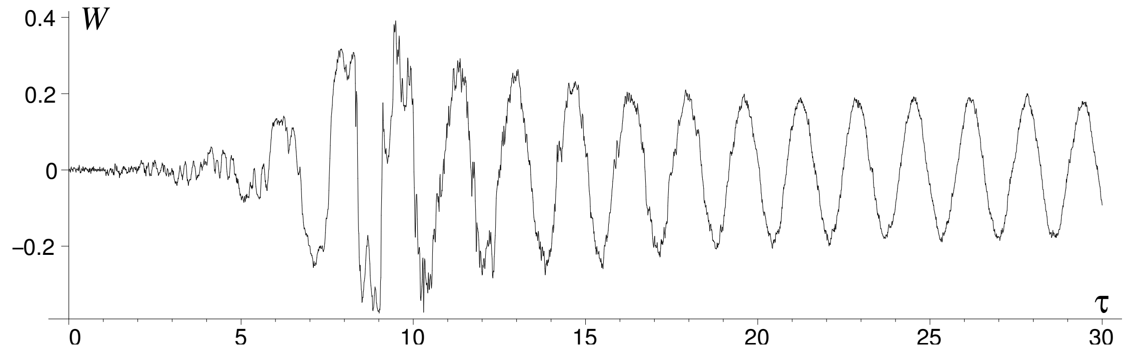

Figure 3: the piston velocity W as a function of time τ. The same run as in previous figure.

Initially, the piston barely moves about its equilibrium position: the hydrodynamic trajectory of the piston is $\bar{Y}(\tau) \equiv 0.5$ for all $\tau > 0$ and this holds exactly (CLS) for $\tau < 2$ as $L \to \infty$. Then, at times τ between 3 and 5, the random vibrations of the piston grow and become quite visible on the y-scale, but for a short while they look random. After that the piston starts travelling back and forth along the y axis, making excursions farther and farther away from the equilibrium point $y = 0.5$. Very soon, at $\tau = \tau_{\text{max}} \approx 8$, the swinging motion of the piston reaches its maximum, $(\Delta Y)_{\text{max}} = \max |Y(\tau) - 0.5| \approx 0.1$.

Then the oscillations of the piston dampen in size and seem to stabilize at an amplitude $A \approx 0.04$ and a period $\tau_{\text{per}} \approx 1.63$. They can be described, in the interval $30 < \tau < 1000$, approximately by

$$Y_1(\tau) \simeq Ae^{-\lambda(\tau-20)} \sin \omega(\tau - \alpha) \qquad (2)$$

with $A = 0.046$ and some constant $\lambda \sim L^{-1/3}$.

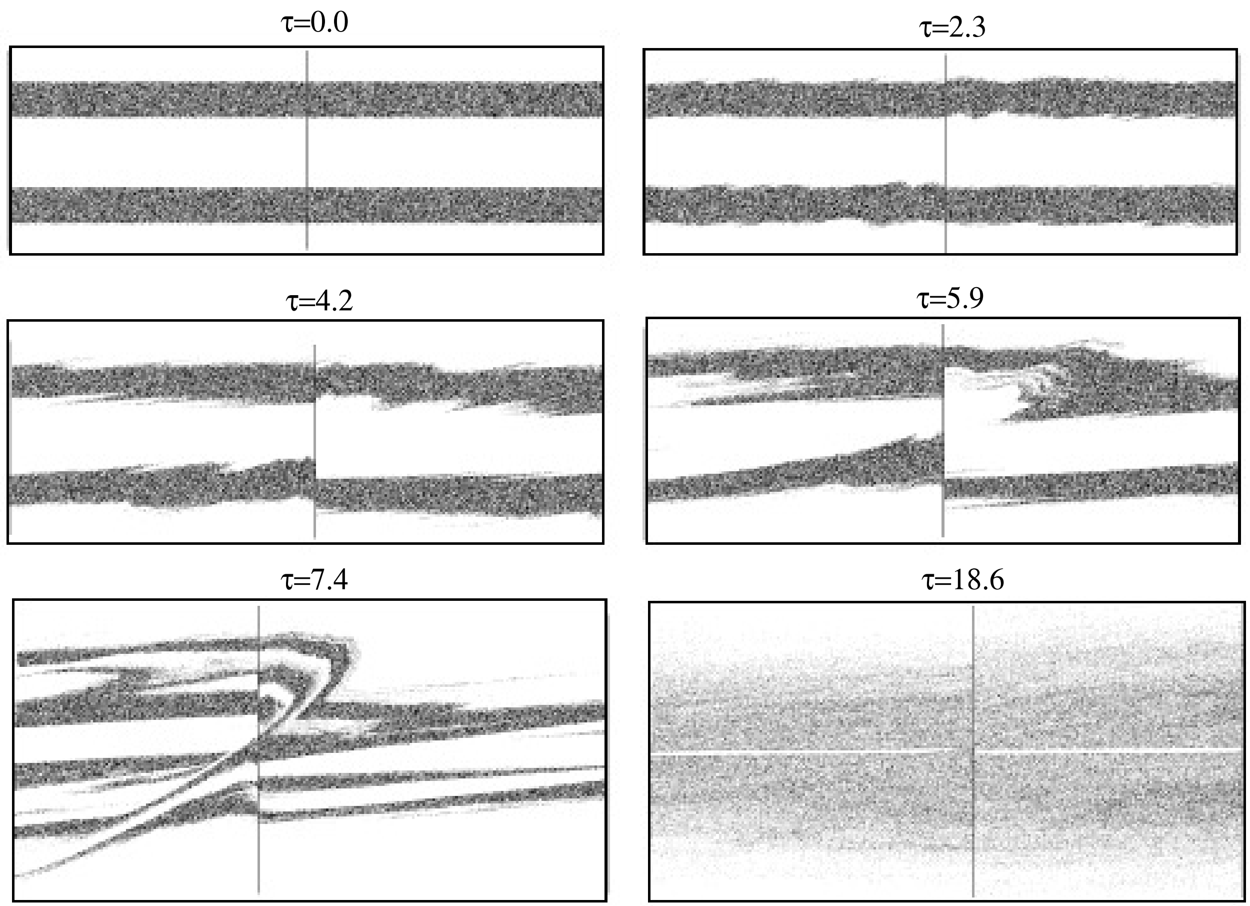

Figure 4: six snapshots of the empirical gas density (in the x, v plane) at times $\tau = 0$, 2.3, 4.2, 5.9, 7.4 and 18.6.

Summary of numerics

Starting with a particular initial particle distribution we found that (1) the velocity distribution of the particles approaches a Maxwellian, i.e. the system goes toward thermal equilibrium on a very long time scale and (2) the time evolution of the piston followed closely, after some initial period, that of a damped harmonic oscillator over an extended time interval, with initial oscillations as large as 1/10 of the system size.

The approach to equilibrium for a system with many degrees of freedom is what we expect on general grounds: almost all of the energy surface consists, for a macroscopic system, of phase points corresponding to equilibrium macrostates. The fact that it takes a very long time can be understood by noting that the amount of energy exchange between a gas particle and the piston is of the order $M^{-1} \sim L^{-2}$ so thermalization takes a long time. But what about the oscillations: can one describe them in some hydrodynamical way?

Hydrodynamical Equations

Lebowitz, Piasecki and Sinai (and some others) derived (heuristically) a closed system of equations that describe the evolution of the system in hydrodynamic variables. This was made rigorous (for very short times) by Chernov, Lebowitz and Sinai who considered the case of an ideal gas: no direct interaction between particles. Denote by $p(y, v, \tau)$ the density of the gas at time τ. The hydrodynamic equations (HE) take the form

$$\left(\frac{\partial}{\partial \tau} + v\frac{\partial}{\partial y}\right) p(y, v, \tau) = 0 \tag{3}$$

(free motion between the piston and the walls),

$$p(0, v, \tau) = p(0, -v, \tau) \qquad \text{and} \qquad p(1, v, \tau) = p(1, -v, \tau) \tag{4}$$

(collisions with the walls) and

$$p(Y(\tau) \pm 0, v, \tau) = p(Y(\tau) \pm 0, 2W(\tau) - v, \tau) \tag{5}$$

(elastic collisions with the piston).

The following is the main equation giving the piston velocity $W(\tau)$:

$$\int (v - W(\tau))^2 \ \mathrm{sgn}(v - W(\tau)) \ q(v, \tau; Y(\tau), W(\tau)) \ dv = 0. \qquad (6)$$

Here $q(v, \tau; Y, W)$ is the density "on the piston" defined by

$$q(v, \tau; Y, W) = \begin{cases} p(Y + 0, v, \tau) & \text{if } v < W \\ p(Y - 0, v, \tau) & \text{if } v > W. \end{cases} \qquad (7)$$

Only the particles that are about to collide with the piston at time τ are taken into account.

The initial conditions are set by $p(y, v, 0) = \pi(y, v)$ and $Y(0)$. Note that $W(0)$ need not be specified, it must satisfy equation (6).

Eq. (6) is essentially a force balance equation—since the rate of collision of the piston with particles on either side and consequent force on piston is much larger than mass of the piston when $N/M \to \infty$, V adjusts instantaneously to make the forces from the two sides balance exactly. The system of (hydrodynamical) equations is now closed and, given initial conditions completely determine the functions $X(t)$, $V(t)$ and $p(x, v, t)$ for $t > 0$. When the initial conditions do not satisfy these equations one has to imagine that they become satisfied instantaneously for $t = 0+$. The HE, like the Vlasov equations for plasmas, are time-reversible. The existence and uniqueness of solutions were proven, under general conditions, in [CLS].

Note switch from y to x and from W to V.

A particular case we considered is when the initial distribution $p(x, v, 0)$ satisfies two conditions:

(S1) *Uniformity and symmetry*. The initial density $p(x, v, 0) = p(x, v)$ is of the form

$$p(x, v) = \begin{cases} p_L(|v|) & \text{for} \quad x < X_0 \\ p_R(|v|) & \text{for} \quad x > X_0 \end{cases}$$

for all v and $X(0) = X_0$.

(S2) *Pressure balance*. The pressure on the piston from both sides is equal:

$$P_L := 2\int_0^\infty v^2 p_L(v)\, dv = P_R := 2\int_0^\infty v^2 p_R(v)\, dv \tag{8}$$

When this happens the system remains frozen in its initial state:

$$X(t) \equiv X_0, \qquad V(t) \equiv 0, \qquad p(x, v, t) \equiv p(x, v, 0),\ \forall t > 0. \tag{9}$$

In work by Caglioti, Chernov and L. (CCL) it was proven that stationary solutions $p(x, v)$ satisfying (S1)–(S2) and an additional monotonicity requirement:

$$p_L(|v_1|) \geq p_L(|v_2|) \quad \text{and} \quad p_R(|v_1|) \geq p_R(|v_2|) \tag{10}$$

for all $|v_1| \leq |v_2|$ are globally stable. This criteria is very similar to the stability criteria for the Vlasov equation described by Penrose, Marchioro and Pulvirenti and by Mouhot and Villani.

Stability analysis

Next CCL analysed the linear stability of the solutions of the HE corresponding to initial densities $p(x,v,0)$ uniform in x across the entire cylinder, i.e. $p(x,v,0) = p_0(|v|)$ for all v and $0 < x < 1$. We also assume that the piston is initially at the midpoint $X(0) = 0.5$. We considered in particular a family of rectangular densities

$$p_0(v) = \begin{cases} 1 & \text{if} \quad r < |v| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $0 < r < 1$ is the parameter of our family. Note that the initial distribution used for the numerical simulations is a particular case of (11) with $r = 1/2$. It turns out that linear stability depends in a very intricate way on r. In particular $r = 1/3$ is linearly stable while $r = 1/2$ is linearly unstable.

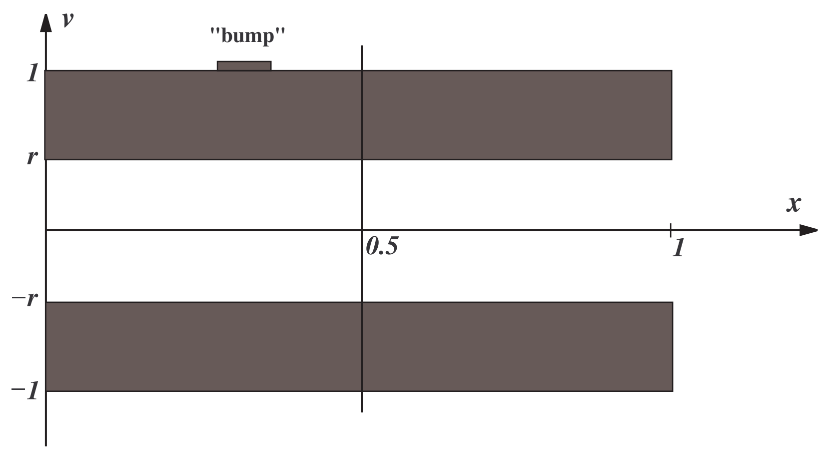

Figure 5: initial rectangular density (11) perturbed by a 'bump'.

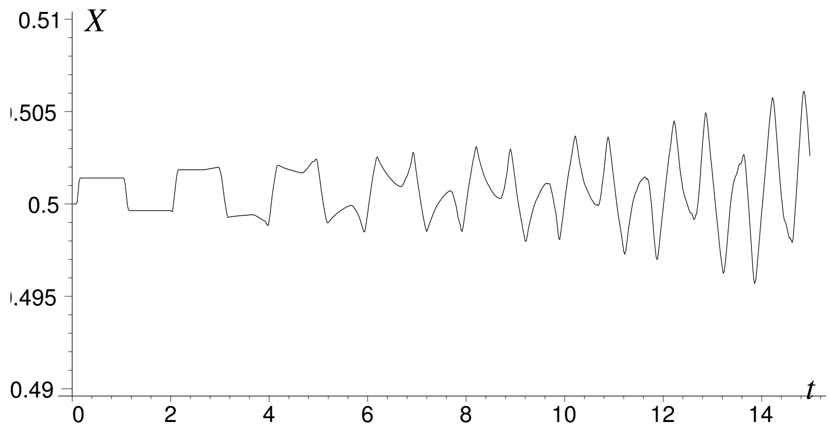

Figure 6: piston's trajectory from the solution of the HE for a perturbed rectangular density with $r = 1/3$ (linearly stable).

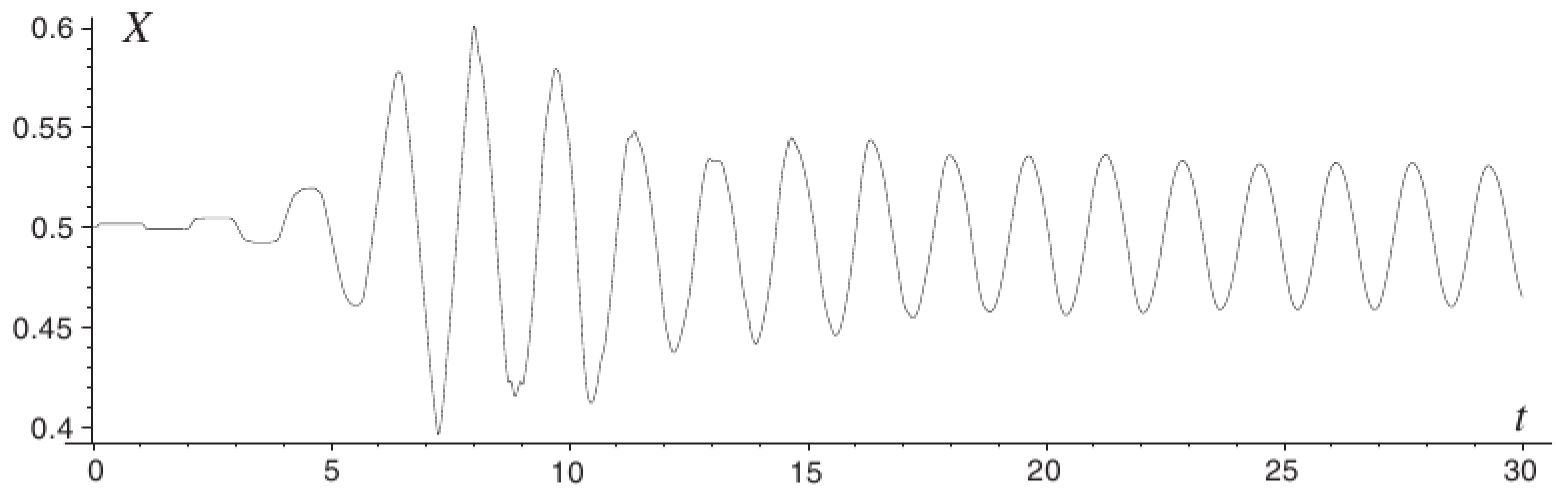

Figure 7: the piston's trajectory from the HE for a perturbed rectangular density with $r = 1/2$ (linearly unstable).

One can see that it behaves almost identically to the simulated trajectory of the piston in figure 7. Thus, not only the initial instability, but also the long term behavior of the simulated piston trajectory match those of perturbed solutions of the hydrodynamical equations.

We also checked on the particle system an initial velocity distribution which is globally stable for the HE and found no changes for very long times. Eventually however the particle velocity distribution did approach a Maxwellian.

An interesting question is the long time behaviour of the HE when the initial $p(v)$ is unstable. It seems like there is a periodic cycle or an invariant manifold of quasi-periodic solutions that acts as an attractor. Of course, due to the time-reversibility of the hydrodynamical equations there can be no attractors in the strict sense. It is more likely that there is an invariant manifold of periodic or quasi-periodic solutions that acts as a saddle point in the phase space: typical trajectories approach that manifold temporarily and then slowly move away. We cannot rigorously prove the existence of periodic or quasi-periodic solutions, but we construct such solutions by using perturbative analysis.

I have currently returned to this problem in the light of the spectacular results for the Vlasov Equation by Mouhot and Villani.