Thermodynamics of the Katok Map

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Dedicated to the memory of Kolya Chernov

Classical Thermodynamic Formalism (Sinai, Ruelle, Bowen)

X is a compact metric space, $f : X \to X$ a continuous map of finite topological entropy, φ a continuous function (potential) on X. Let $M(f)$ be the space of all f-invariant Borel probability ergodic measures on X. A measure $\mu_\varphi \in M(f)$ is an equilibrium measure if

$$P(\varphi) := \sup_{\mu \in M(f)} \{h_\mu(f) + \int_X \varphi \, d\mu\} = h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi,$$

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Of special interest is the geometric potential: a family of potential functions $\varphi_t(x) = -t \log |\mathrm{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. The pressure function $P(t) := P(\varphi_t)$ is real analytic in t.

Continuous potentials. Phase Transitions

The classical case can be extended in two directions: 1) to maps that are not uniformly hyperbolic and 2) to potentials that are not Hölder continuous. Either way one can experience phase transitions, i.e., absence or non-uniqueness of equilibrium measures.

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Consider the map $f(x) = 2x \pmod 1$ on $I = [0, 1]$ and the potential function $\varphi(x) = c(1 - \log x)^\alpha$, which is continuous everywhere and is smooth everywhere but at 0 where it is not even Hölder continuous.

Theorem (Sarig and Zhang-P.)

1. *For any $\alpha > 1$ and any c, there is a unique equilibrium measure μ_φ for φ such that $\mu_\varphi((0,1]) = 1$;*
2. *For any $0 < \alpha \leq 1$ there is a constant $c_0 < 0$ such that*
   - *for $c_0 < c < 0$ there is a unique equilibrium measure μ_φ for φ such that $\mu_\varphi((0,1]) = 1$;*
   - *for $c < c_0$ there is no equilibrium measures for φ such that $\mu_\varphi((0,1]) = 1$; the Dirac measure at 0 is the equilibrium measure.*

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The proof of this theorem involves constructing a countable Markov partition of $(0,1]$ by intervals $I_n = [\frac{1}{2^n}, \frac{1}{2^{n+1}}]$ such that f is conjugate to the renewal shift (Σ_A, σ) where the transition matrix A allows only pairs $(0,n)$ and $(n, n-1)$, $n \geq 1$. One can now use the powerful techniques by Sarig to establish existence and uniqueness of equilibrium measures for renewal shifts.

Little is known on existence and uniqueness of equilibrium measures outside the uniform hyperbolicity case (except for SRB measures) but this is an emerging area of research:

1. one-dimensional maps:
   - unimodal and multimodal maps (Bruin-Todd, P.-Senti);
   - maps with indifferent fixed points, e.g., the Manneville-Pomeau map (Pollicott-Weiss, Sarig, Hu).
2. rational maps (Przytycki-Letelier, Makarov-Smirnov);
3. geodesic flows on rank one manifolds of non-positive curvature (Burns-Climenhaga-Fisher-Thompson);
4. “nonuniform specification” (Climenhaga, Thompson).

The Katok Map (A. Katok, 1979)

Consider the automorphism of the two-dimensional torus $\mathbb{T}$ given by the matrix $T := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

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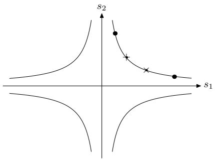
Consider the automorphism of the two-dimensional torus $\mathbb{T}$ given by the matrix $T := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let $D_r = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \leq r^2\}$ where (s_1, s_2) is the coordinate system obtained from the eigendirections of T. Choose small $r_0 > 0$ and set $r_1 = (\log \lambda) r_0$ where $\lambda > 1$ is the eigenvalue of T. We have $D_{r_0} \subset \operatorname{Int} T(D_{r_1}) \cap \operatorname{Int} T^{-1}(D_{r_1})$.

In D_{r_0} the map T is the time-1 map of the flow generated by the system of ODE:



$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda,$$

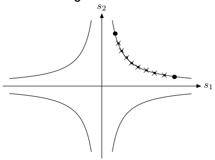
where $\lambda > 1$ is the eigenvalue of T.

Choose a function $\psi : [0,1] \mapsto [0,1]$ satisfying:

1. ψ is C^∞ everywhere except at the origin;
2. $\psi(u) = 1$ for $u \geq r_0$ and some $0 < r_0 < 1$;
3. $\psi'(u) > 0$ and is decreasing for every $0 < u < r_0$;
4. $\psi(u) = (u/r_0)^\alpha$ for $0 \leq u \leq \frac{r_0}{2}$ and some $0 < \alpha < 1$.

We slow down trajectories by perturbing the flow in D_{r_0} as follows

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$
$$\dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$



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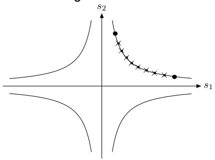


This system of equations generates a local flow and let g be the time-1 map of this flow. The choices of ψ and r_0 guarantee that the domain of g contains D_{r_0}. Furthermore, g is of class C^∞ except at the origin and it coincides with T in some neighborhood of the boundary ∂D_{r_0}. Therefore, the map

$$G(x) = \begin{cases} T(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_0}, \\ g(x) & \text{if } x \in D_{r_0} \end{cases}$$

defines a homeomorphism of the torus $\mathbb{T}^2$, which is a C^∞ diffeomorphism everywhere except at the origin.

Since $0 < \alpha < 1$, we obtain that

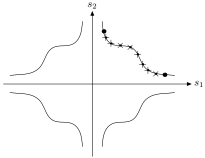
$$\int_0^1 \frac{du}{\psi(u)} < \infty.$$

This implies that the map G preserves the probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$, where m is the area and the density κ is a positive C^∞ function that is infinite at 0 and is defined by

$$\kappa(s_1, s_2) := \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1, s_2) \in D_{r_0}, \\ 1 & \text{otherwise} \end{cases}$$

and $\kappa_0 := \int_{\mathbb{T}^2} \kappa\, dm$.

The map g can be further perturbed to a map F that preserves area and is of class C^∞.



This can be done by a coordinate change ϕ in $\mathbb{T}^2$. Define ϕ in D_{r_0} by the formula

$$\phi(s_1, s_2) := \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2+{s_2}^2} \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2)$$

and set $\phi = \mathrm{Id}$ in $\mathbb{T}^2 \setminus D_{r_0}$. Clearly, ϕ is a homeomorphism and is a C^∞ diffeomorphism outside the origin. One can show that ϕ transfers the measure ν into the area, and that $F = \phi \circ G \circ \phi^{-1}$ is a C^∞ diffeomorphism called the Katok map.

Properties Of The Katok Map

1. F is topologically conjugate to T by a homeomorphism H.
2. There exist two continuous, uniformly transverse, invariant one-dimensional stable E^s and unstable E^u distributions.
3. For almost every $x \in \mathbb{T}^2$ the Lyapunov exponent $\chi(x, v) > 0$ for $v \in E^u(x)$ and $\chi(x, w) < 0$ for $w \in E^s(x)$.
4. F has two continuous, uniformly transverse, invariant one-dimensional foliations with smooth leaves. They are stable W^s and unstable W^u foliations for F and are the images under H of the stable and unstable foliations for T respectively.
5. for every $\varepsilon > 0$ there is $r_0 > 0$ such that
$$|\int \log|\mathrm{Jac}(dF|E^u(x)|\, dm - \log\lambda| < \varepsilon.$$
6. F lies on the boundary of Anosov diffeomorphisms.

The Main Theorem

Let F be the Katok map. Consider the geometric potential $\varphi_t(x) = -t \log |\mathrm{Jac}(dF|E^u(x))|$, which is continuous in x.

Theorem (P., Senti, Zhang)

The following statements hold:

1. *For any $t_0 < 0$, one can choose a sufficiently small $r_0 = r_0(t_0)$ such that for every $t_0 < t < 1$*
   - *there exists a unique equilibrium ergodic measure μ_t;*
   - *μ_t has exponential decay of correlations and satisfies the CLT with respect to a class of potential functions which includes all Hölder continuous functions on $\mathbb{T}^2$;*
2. *The pressure function $P(t)$ is real analytic in $t \in (t_0, 1)$;*
3. *For $t = 1$ there are two equilibrium measures associated to φ_1, namely the Dirac measure at the origin and the area. The latter has polynomial decay of correlations.*
4. *For $t > 1$ the Dirac measure at the origin is the unique equilibrium measure associated to φ_t.*

Problems

- Is it true that given a number r_0, the pressure function $P(t)$ for the Katok map F is real analytic for all $-\infty < t < 1$, i.e., there is no phase transitions for this set of t?
- Is it true that given a number r_0, the pressure function $P(t)$ for the Katok map F, computed over the space of invariant measures with zero weight at the origin, is real analytic for all $-\infty < t < \infty$, i.e., there is no phase transitions within this class of invariant measures?

An Inducing Scheme For Automorphism T

Consider a finite Markov partition for the automorphism T and let $\tilde{P}$ be a partition element which lies "far away" from the origin. $\tilde{P}$ is a rectangle with direct product structure generated by full length stable $\tilde{\gamma}^s(x)$ and unstable $\tilde{\gamma}^u(x)$ curves (i.e., segments of stable and unstable lines). Moreover, given $\varepsilon > 0$, we can choose the Markov partition such that its diameter is $\leq \varepsilon$.

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For $x \in \tilde{P}$ let $\tilde{\tau}(x)$ be the first return time of x to Int $\tilde{P}$. For all x with $\tilde{\tau}(x) < \infty$, the connected component of the level set $\tilde{\tau} = \tilde{\tau}(x)$ containing x is an *s*-set

$$\tilde{\Lambda}^s(x) = \bigcup_{y \in \tilde{U}^u(x) \setminus \tilde{A}^u(x)} \tilde{\gamma}^s(y),$$

where $\tilde{U}^u(x) \subseteq \tilde{\gamma}^u(x)$ is an interval containing x and $\tilde{A}^u(x) \subset \tilde{U}^u(x)$ is the set of points which either lie on the boundary of the Markov partition or never return to the set $\tilde{P}$. Note that $\tilde{A}^u(x)$ has zero Lebesgue measure (length) in $\tilde{\gamma}^u(x)$.

We thus obtained the collection $\tilde{S}$ of basic sets $\{\tilde{J}_i := \tilde{\Lambda}_i^s\}$ and numbers $\{\tilde{\tau}_i\}$. They generate an inducing scheme $\{\tilde{S}, \tilde{\tau}\}$ for T with

- $\tilde{W} := \bigcup_i \tilde{J}_i \subset \tilde{P}$ – the inducing domain;
- $\tilde{\tau}(x) := \tilde{\tau}_i$ for $x \in \tilde{J}_i$ – the inducing time;
- $\tilde{\mathcal{F}}(x) := F^{\tilde{\tau}(x)}(x)$, $x \in \tilde{J}_i$ – the induced map.

An Inducing Scheme For The Katok Map F

Applying the conjugacy map H, we obtain the collection S of basic sets $\{J_i := H(\tilde{J}_i) = H(\tilde{\Lambda}_i^s)\}$ and numbers $\{\tau_i = \tilde{\tau}_i\}$ that generate an inducing scheme $\{S, \tau\}$ for F with

- $W := \bigcup_i J_i \subset P := H(\tilde{P})$ – the inducing domain;
- $\tau(x) := \tau_i$ for $x \in J_i$ – the inducing time;
- $\mathcal{F}(x) := F^{\tau(x)}(x)$, $x \in J_i$ – the induced map.

Observe that P is an element of the Markov partition for F and that $J_i(x)$ is an s-set for F, i.e.,

$$J_i(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

where γ^s is the full length stable curve, $U^u(x) \subseteq \gamma^u(x)$ (a full length unstable curve) is an open set containing x and $A^u(x) \subset U^u(x)$ is the set of points which either lie on the boundary of the Markov partition or never return to the set P. Note that $A^u(x)$ has zero Lebesgue measure in $\gamma^u(x)$.

The inducing scheme $\{S, \tau\}$ has the following properties:

(H1) Markov property: for any $J \in S$ we have
$F^{\tau(J)}(J) \subset W \quad \text{and} \quad \bigcup_{J \in S} F^{\tau(J)}(J) = W.$
Note that $F^{\tau(J)}(J)$ is a u-set which consists of full length unstable curves.

(H2) Generating property: the partition of the inducing domain W by basic sets J is two-sided generating; more precisely, for every bi-infinite sequence $\underline{a} = (a_n)_{n \in \mathbb{Z}} \in S^{\mathbb{Z}}$ there exists a *unique* sequence $\underline{x} = \underline{x}(\underline{a}) = (x_n = x_n(\underline{a}))_{n \in \mathbb{Z}}$ such that

$$x_n \in \overline{J_{a_n}} \quad \text{and} \quad F^{\tau(J_{a_n})}(x_n) = x_{n+1}.$$

These conditions allow one to obtain a symbolic representation of the induced map $\mathcal{F}$ as the Bernoulli shift σ on the countable set of states $S^{\mathbb{Z}}$ via the coding map π given by $\pi(\underline{a}) := x_0(\underline{a})$. This map is well defined, continuous and

- $\pi \circ \sigma(\underline{a}) = \mathcal{F} \circ \pi(\underline{a})$ for all $\underline{a} \in S^{\mathbb{Z}}$;
- π is one-to-one on the set

$$\check{S} := \{\underline{a} \in S \colon x_n(\underline{a}) \in J_{a_n} \text{ for all } n \in \mathbb{Z}\}$$

  and $\pi(\check{S}) = W$.

The inducing scheme $\{S, \tau\}$ has two additional crucial properties:

(H3) the set $S^{\mathbb{Z}} \setminus \check{S}$ supports no shift-invariant measure which gives positive weight to any open subset.

This property ensures that every Gibbs measure for σ is supported on the set $\check{S}$ and thus its projection by π is supported on W and is invariant under the induced map $\mathcal{F}$. This projection is a natural candidate for the equilibrium measure for F.

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(H4) there exists $h_1 < h_{\text{top}}(T)$ such that

$$S_n := \text{Card}\,\{J \in S \colon \tau(J) = n\} \leq e^{h_1 n}.$$

This property provides an exponential bound for the number of basic elements with the given inducing time.

Since the conjugacy H preserves combinatorial information about T, the inducing schemes for F and T have the same number S_n of basic elements with inducing time $\tilde{\tau}_i = n$.
Hence, it suffices to estimate the number of sets $\tilde{\Lambda}_i^s$ with a given i. This number is less than the number of periodic orbits of T that originate in $\tilde{P}$ and have minimal period $\tilde{\tau}_i$. Using the symbolic representation of T as a subshift of finite type, one can see that the latter equals the number of symbolic words of length $\tilde{\tau}_i$ for which the symbol $\tilde{P}$ occurs only as the first and last symbol (but nowhere in between). The number of such words is known to grow exponentially with some exponent $h_1 < h_{\mathrm{top}}(T)$.

Young's Diffeomorphisms (L.-S. Young, 1998)

A map f is a Young's diffeomorphism if

(Y0) There exists $\Lambda \subset M$ with hyperbolic product structure generated by stable and unstable curves γ^s and γ^u such that $\mu_{\gamma^u}(\gamma^u \cap \Lambda) > 0$ for all $\gamma^u \in \Gamma^u$, where μ_{γ^u} is the leaf volume on γ^u.

(Y1) There exists a countable collection of continuous subfamilies $\Gamma_i^s \subset \Gamma^s$ of stable curves and positive integers τ_i such that the s-subsets $\Lambda_i^s := \bigcup_{\gamma \in \Gamma_i^s} (\gamma \cap \Lambda) \subset \Lambda$ are pairwise disjoint and satisfy

1. invariance: for every $x \in \Lambda_i^s$
$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)), \; f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x));$$
2. Markov property: $\Lambda_i^u := f^{\tau_i}(\Lambda_i^s)$ is a u-subset of Λ such that
$$f^{-\tau_i}(\gamma^s(f^{\tau_i}(x)) \cap \Lambda_i^u) = \gamma^s(x) \cap \Lambda,$$
$$f^{\tau_i}(\gamma^u(x) \cap \Lambda_i^s) = \gamma^u(f^{\tau_i}(x)) \cap \Lambda.$$

(Y2) For every $\gamma^u \in \Gamma^u$ we have $\mu_{\gamma^u}((\Lambda \setminus \cup \Lambda_i^s) \cap \gamma^u) = 0$.

For any $x \in \Lambda_i^s$ define the *inducing time* by $\tau(x) := \tau_i$ and the *induced map* $\mathcal{F} : \bigcup \Lambda_i^s \to \Lambda$ by $F|_{\Lambda_i^s} := f^{\tau_i}|_{\Lambda_i^s}$.

(Y3) There exists $0 < \alpha < 1$ such that
  (a) $d(\mathcal{F}(x), \mathcal{F}(y)) \le \alpha\, d(x, y)$ for $x \in \Lambda_i^s$ and $y \in \gamma^s(x)$;
  (b) $d(x, y) \le \alpha\, d(\mathcal{F}(x), \mathcal{F}(y))$ for $x \in \Lambda_i^s$ and $y \in \gamma^u(x) \cap \Lambda_i^s$.

(Y4) The induced map $\mathcal{F}$ has the bounded distortion property;

(Y5) There exists $\gamma^u \in \Gamma^u$ such that

$$\sum_{i=1}^{\infty} \tau_i \mu_{\gamma^u}(\Lambda_i^s) < \infty.$$

Thermodynamics Of Young's Diffeomorphisms

The following general result describes existence, uniqueness and ergodic properties of Young's diffeomorphisms for the geometric potential φ_t.

Theorem (P., Senti, Zhang)

The following statements hold:

1. *There exists an equilibrium measure μ_1 for the potential φ_1 which is a unique SRB measure;*
2. *Assume that the inducing scheme $\{S, \tau\}$ for f satisfies $S_n \le e^{h_1 n}$ with $0 < h_1 < -\int \varphi_1 \, d\mu_1$. Then there is a number $t_0 < 0$ such that for every $t_0 < t < 1$ there exists a measure μ_t which is a unique equilibrium measure for the potential φ_t;*
3. *μ_t has exponential decay of correlations and satisfies the CLT with respect to a class of potential functions which contains all Hölder continuous functions on M.*

The proof of this theorem produces an explicit formula for the number t_0, namely

$$t_0 := \frac{h_1 + \lambda_1}{\lambda + \lambda_1},$$

where $\lambda_1 = \int \varphi_1 \, d\mu_1 < 0$ and

$$\lambda = \max_{x \in W} \log \operatorname{Jac}(d\mathcal{F}|E^u(x)) > 0.$$

One can show that $\lambda + \lambda_1 > 0$. Since by our assumptions, $h_1 + \lambda_1 < 0$, this ensures that $t_0 < 0$. Furthermore, $|t_0|$ can be made arbitrarily large if one arranges for λ to be sufficiently close to $-\lambda_1$. This can be done for the Katok map.

Observe that $h_1 < -\int \varphi_1 \, d\mu_1$. Indeed, it holds for the automorphism T, and hence, for the Katok map F provided r_0 is sufficiently small. Thus, there is a number $t_0 < 0$ such that for every $t_0 < t < 1$ one can construct a unique equilibrium measure μ_t associated to the geometric potential φ_t among all the measures μ for which $\mu(P) > 0$. Note that $\mu_t(U) > 0$ for every open set $U \subset P$.

Proof Of The Main Theorem

Observe that $h_1 < -\int \varphi_1 \, d\mu_1$. Indeed, it holds for the automorphism T, and hence, for the Katok map F provided r_0 is sufficiently small. Thus, there is a number $t_0 < 0$ such that for every $t_0 < t < 1$ one can construct a unique equilibrium measure μ_t associated to the geometric potential φ_t among all the measures μ for which $\mu(P) > 0$. Note that $\mu_t(U) > 0$ for every open set $U \subset P$.
Let us choose another element $\hat{P}$ of the Markov partition which is far away from the origin. Repeating the above argument one can find a number $\hat{t}_0$ and construct a measure $\hat{\mu}_t$ which is a unique equilibrium for the geometric potential φ_t with $\hat{t}_0 < t < 1$ among all measures μ for which $\mu(\hat{P}) > 0$. Furthermore, we have that $\hat{\mu}_t(\hat{U}) > 0$ for every open set $\hat{U} \subset \hat{P}$.

Since the map f is topologically transitive, for any open sets $U \subset P$ and $\hat{U} \subset \hat{P}$ there exists k such that $f^k(U) \cap \hat{U} \neq \emptyset$. Therefore, $\mu_t = \hat{\mu}_t$.

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If the number r_0 in the construction of the Katok map is sufficiently small, the union of partition elements that are far away from the origin form a closed set whose complement is a neighborhood of zero. Observe that the only measure which does not charge any element of the Markov partition that lies outside this neighborhood is the Dirac measure at the origin.

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Finally, using again the above explicit formula for the number t_0, one can show that the sum $\lambda_1 + \lambda$ (where $\lambda_1 = \lambda_1(r_0)$ and $\lambda = \lambda(r_0)$) will become arbitrarily close to zero and hence, the number t_0 can be made arbitrarily large, if the number r_0 in the construction of the Katok map is sufficiently small.