Stochastic Perturbations of Convex Billiards

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We consider a strictly convex billiard table with C^2 boundary. Each time the billiard ball hits the boundary its reflection angle has a small random perturbation.

The perturbation distribution corresponds to the physical situation where the scale of the surface irregularities is smaller than but comparable to the diameter of the reflected object.

We prove that for a large class of such perturbations the resulting Markov chain is uniformly ergodic.

Stochastic properties proved via topological dynamics methods

I would like to thank Ya. G. Sinai for suggesting this problem.

Billiards with stochastic perturbation are very natural models motivated by microscopic kinetic problems, theoretical computer science, etc. Have been receiving increased attention from deterministic and stochastic dynamics communities in the last decade.

In most of the studied cases, the outgoing angle is either uniformly distributed, or satisfying other reflection laws that are physically relevant if the micro-structure and irregularities of the boundaries have a length-scale larger than the diameter of the ball.

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We focus on the case of stochastic perturbations corresponding to the physical situation where the scale of the surface irregularities is smaller than but comparable to the diameter of the reflected object

Deterministic billiards on sufficiently smooth strictly convex tables are non-ergodic

In contrast to the deterministic situation, for a certain class of stochastic perturbations of a reflection law, the associated Markov process is uniformly ergodic, and any probability measure converges exponentially fast to a unique invariant probability measure.

We note that this is not true in general: there are examples of stochastic perturbations under which the resulting system is not ergodic.

Our result holds for billiard tables which are strictly convex, with C^2 boundary, including the possibility of isolated points of null curvature.

Informal description of the mathematical setup.

Given a prescribed family of independent random variables $\{Y_\theta\}_{\theta\in[0,\pi]}$, the dynamics obeys a stochastic rule. If the outgoing angle after a deterministic collision would have been θ, it is taken as $\theta + Y_\theta$ instead.

The family $\{Y_\theta\}_{\theta\in[0,\pi]}$ is chosen in such a way that typically the influence of Y_θ is negligible compared to θ. However, it becomes substantial when the incidence angle gets too small. The latter property reflects an increased sensitivity to surface rugosity.

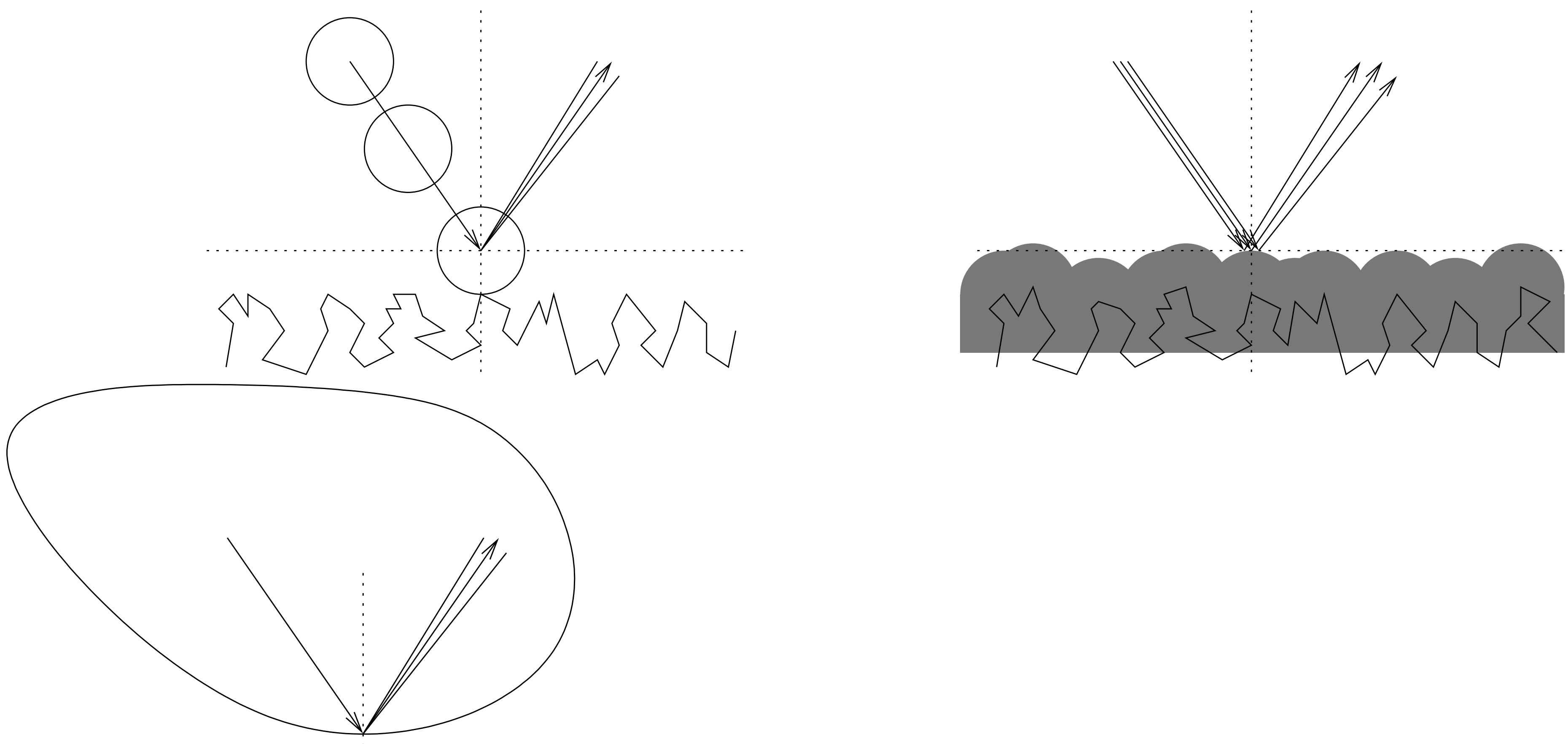

Figure 1: Round particle reflecting on a rough surface; equivalent microscopic model of point particle reflecting on a smooth surface; macroscopic model.

Models and result

Description of deterministic billiard in D.

We assume that D is a connected domain in $\mathbb{R}^2$, strictly convex with C^2 boundary.

Notice that isolated points with null curvature are allowed.

The billiard in D is the dynamical system describing the free motion of a point mass inside D with elastic reflections at its boundary Γ. Let $n(q)$ be the unit normal to the curve Γ at the point q pointing towards the interior of D. The phase space of such a dynamical system is $\{(q,v) : q \in \Gamma, |v| = 1, \langle v, n(q)\rangle \geqslant 0\}$.

The image of a point (q_0, v_0) by the deterministic billiard map T is denoted by

$$T(q_0, v_0) = (q_1, v_1)$$

We take the set of coordinates (s, θ),
s: arc-length parameter along Γ and
$\theta \in [0, \pi]$: angle between the oriented tangent line to the boundary at q and v.

The phase space is given by the cylinder

$$M = \{(s, \theta) : 0 \leqslant s < |\Gamma|, 0 \leqslant \theta \leqslant \pi\}.$$

For $x = (s, \theta) \in M$, we write $s(x) = s, \theta(x) = \theta$, and also $q(x)$ for the corresponding point in Γ.

T is a diffeomorphism defined on compact set M with fixed points at $\partial M = \{(s, \theta) : \theta = 0 \text{ or } \pi\}$.

Moreover, T is a *twist diffeomorphism*: the image of any vertical line ($s =$ constant) is a smooth curve with slope positive and bounded away from infinity.

If D is strictly convex with sufficiently smooth boundary, by KAM theory there exist invariant curves of the billiard map as close as we want to the boundary ∂M Therefore, if the initial angle is small it remains small along the whole trajectory. This regularity can be broken using arbitrarily small random perturbations.

We consider the system with random perturbations that act on the outgoing angle, independently of the position, by adding a random variable to θ.

Fix $0 < \epsilon < \frac{\pi}{2}$. For θ away from 0 and π, we take the probability density of the outgoing angle as a constant on the interval $[\theta-\epsilon, \theta+\epsilon]$ (the correct law derived from Figure 1 would not have constant density, but this is irrelevant for the qualitative behavior of the model).

The particular choice of perturbation becomes more delicate when the collision angle is close to the extremes.

Three examples to illustrate possible behaviors.

1. For every point $x = (s, \theta) \in M$, define $\theta^\epsilon := \min\{\max(\theta, \epsilon), \pi - \epsilon\}$; consider the measure Q_x^ϵ on M:

$$Q_x^\epsilon(A) = \int_{\theta^\epsilon - \epsilon}^{\theta^\epsilon + \epsilon} I_A(s, u) \frac{1}{2\epsilon} \mathrm{d}u.$$

In other words, the random outgoing angle is distributed uniformly on

$$[\theta^\epsilon - \epsilon, \theta^\epsilon + \epsilon].$$

2. Outgoing angle uniformly distributed on

$$[\max\{\theta - \epsilon, 0\}, \min\{\theta + \epsilon, \pi\}]$$

3. Outgoing angle uniformly distributed on $[0, 2\theta]$ for $\theta < \epsilon$, and defined analogously for $\theta > \pi - \epsilon$.

In Examples 1 and 2, the outgoing angle is uniformly distributed over an interval whose length is at least ϵ for $\theta < \epsilon$.

Example 1 replaces θ by a uniform on $[0, 2\epsilon]$, whereas

Example 2 replaces θ by a uniform on $[0, \theta + \epsilon]$.

Example 3, on the other hand, is different in nature.

For a trajectory where the outgoing angle would be almost tangent to the boundary Γ, the replacement by a uniform on $[0, 2\theta]$ keeps it very close to being tangent. We find the first two examples natural to justify for a physical situation of a rigid sphere hitting a rough surface, as shown in Figure 1.

We concentrate on Example 1, bearing in mind that all the arguments translate seamlessly to Example 2 or other similar cases.

$\mathcal{P}$: set probability measures on $\mathcal{B}$; total variational distance on $\mathcal{P}$: $\|\mu - \nu\| = \sup_{A\in\mathcal{B}} |\mu(A) - \nu(A)|$.

Stochastic perturbation of T given by transition kernel $P_\epsilon(x, A) = Q^\epsilon_{Tx}(A),\ x \in M,\ A \in \mathcal{B}$.

$P_\epsilon(., A)$ is a measurable function for every $A \in \mathcal{B}$, and $P_\epsilon(x, .)$ is a measure on $\mathcal{B}$ for every $x \in M$.

Def. Push-forward operator $\mu \mapsto \mu P_\epsilon$ for $\mu \in \mathcal{P}$:

$$\mu P_\epsilon(A) = \int_M \mu(\mathrm{d}x) P_\epsilon(x, A);$$

Def. $\mu \in \mathcal{P}$ is *invariant for* P_ϵ if $\mu P_\epsilon = \mu$.

Theorem 1. *Suppose that D is strictly convex and its boundary Γ is C^2. For each $0 < \epsilon < \frac{\pi}{2}$, there exists a unique invariant measure ν_ϵ for P_ϵ, and moreover there exists $\gamma > 0$ such that*

$$\left\| \mu P_\epsilon^n - \nu_\epsilon \right\| \leqslant e^{-\gamma n} \text{ for all } \mu \in \mathcal{P} \text{ and } n \in \mathbb{N}.$$

Theorem 1 remains valid, with essentially the same proof, for a much broader class of distributions. What is important is that the probability density of the outgoing angle is bounded from below on some interval around θ whose length is also bounded from below.

Stochastic dynamics constructed using the distribution of Example 3 is not only non-ergodic, but it gets quickly absorbed by a random point at the boundary ∂M.

Deterministic billiards in convex tables

T preserves prob. measure $d\nu = \frac{1}{2|\Gamma|} \sin\theta \, ds \, d\theta$.

Lemma 2. *If* $T(x) = (s_1(x), \theta_1(x))$, $x = (s, \theta)$, *then* $\left(\frac{\partial s_1}{\partial \theta}\right)^{-1}$ *can be continuously extended to the boundary of* M. *In particular,* $\frac{\partial s_1}{\partial \theta}$ *is bounded away from zero.*

T being a twist map holds if D is convex with C^1 boundary. In this case the map T is an homeomorphism in $\mathrm{int} M$ that can be extended defining $Tx = x$ for every $x \in \partial M$. $x \mapsto q(T(x))$ is not continuous in x if $q(x)$ is in the interior of a segment of the boundary, but $x \mapsto \theta(T(x))$ is nonetheless continuous.

For any $0 < a < \frac{\pi}{2}$, we define the cylinder $M_a = [0, |\Gamma|) \times [a, \pi - a]$.

Lemma 3. *Suppose that D is convex with C^1 boundary. Given $\epsilon > 0$, there exist $0 < c_2 < c_1 < \epsilon$ satisfying the following conditions: $M_\epsilon \subset T(M_{c_1})$, $M_{c_1} \subset T^2(M_{c_2})$ and $T^2(M_{c_1}) \subset M_{c_2}$.*

For the deterministic billiard on a sufficiently smooth table, Lazutkin regularity result:

If D is convex with smooth boundary and curvature bounded from below then there exists $M' \subset M$ with positive measure, foliated by invariant curves; M' accumulates on the horizontal boundaries of M, the map T restricted to each such curve is topologically equivalent to an irrational rotation; close to the boundary ($\theta = 0$ or π in the phase space) there is a set of positive measure with regular behavior.

Theorem 1 shows that this regularity can be broken by an arbitrarily small stochastic perturbation.

Examples of convex regions with no invariant curves near the boundary. Have trajectories with an infinite number of bounces in finite time. Either violating the condition on the curvature or the differentiability of the boundary.

Halpern: curve that has nowhere vanishing curvature but unbounded third derivative, proved that there are trajectories baring this pathological behavior.

Mather: convex billiard with C^2 boundary violating the condition of non-null curvature, it has trajectories coming arbitrarily close to being positively tangent to the boundary

Markov chains and their densities

Recall that the stochastic perturbation of the map T is given by the transition kernel

$$P_\epsilon(x, A) = Q^\epsilon_{Tx}(A), \quad x \in M, \ A \in \mathcal{B}.$$

Let P^n_ϵ denote the n-th power of the kernel

$$P^{n+1}_\epsilon(x, A) = \int_M P_\epsilon(x, \mathrm{d}y) P^n_\epsilon(y, A).$$

$P^n_\epsilon(x, .)$ is a probability measure on $\mathcal{B}$ for every $x \in M$. Moreover, as operators on $\mathcal{P}$ they satisfy $\mu P^n_\epsilon = (\mu P^{n-1}_\epsilon) P_\epsilon$, and defining $P^0_\epsilon(x, A) = I_A(x)$, the set $(P^n_\epsilon)_{n \in \mathbb{N}_0}$ forms a semi-group.

Proposition 4. *For the stochastic billiard map there exist density functions* $p_\epsilon^n(x, y)$ *such that, for every* $x = (s, \theta) \in M$, $n \geqslant 2$, *and* $A \in \mathcal{B}$,

$$P_\epsilon^n(x, A) = \int_A p_\epsilon^n(x, y)\mathrm{d}y.$$

Proof. If $A = [\tilde{s}, \hat{s}] \times [\tilde{\theta}, \hat{\theta}], Tx = (s_1, \theta_1), z = (s_1, \theta')$, $Tz = (s_1', \theta_1')$ then, for $z \in \hat{T}x(s' = s_1)$, we have

$$P_\epsilon^2(x, A) = \frac{1}{4\epsilon^2} \int_{[\theta_1^\epsilon - \epsilon, \theta_1^\epsilon + \epsilon]} d\theta' \int_{[-\epsilon, \epsilon]} I_A(s_1', \theta_1'^\epsilon + u)\mathrm{d}u.$$

Changing variables $d\theta' = \frac{\partial \theta'}{\partial s_1'}\, ds_1'$, we obtain the desired density. The general case is analogous. $\square$

A Markov chain is said to satisfy *Döblin's condition* if there exists a probability measure λ, $m > 0$ and $\delta_1 < 1$, $\delta_2 > 0$ such that, whenever $\lambda(A) > \delta_1$, then $P_\epsilon^m(x, A) > \delta_2$, for all $x \in M$.

Theorem 1 is a consequence of the following result.

Proposition 5. *Suppose that D is strictly convex and its boundary Γ is C^2. Then for every $0 < \epsilon < \pi/2$, there exist $b > 0$ and $N > 0$ such that*

$$p_\epsilon^N(x, y) > b, \quad \forall\, x, y \in M.$$

Proof of Theorem 1. Proposition 5 implies that the chain is ψ-irreducible and satisfies Döblin's condition; then it is uniformly ergodic, The result then follows from general results included in the book by Meyn and Tweedie: Markov Chains and Stochastic Stability. □

Proof of Döblin's condition.

Before proving Proposition 5 and Theorem 1, we need a few additional technical steps, summarized in the next definitions and two propositions.

We say that a sequence $\xi = (\xi_k)_{k\in[0,\dots,l]}$, $\xi_k = (s_k, \theta_k) \in M$ is *an ϵ-angular perturbed orbit of length l* if, for all $0 \leqslant k < l$, $s(T(\xi_k)) = s_{k+1}$ and $|\theta(T(\xi_k)) - \theta_{k+1}| < \epsilon$.

Let $\mathcal{O}_{\epsilon,l}$ denote the set of ϵ-angular perturbed orbits of length l.

For $n > 0$ define

$$\widehat{T}^n_\epsilon(x) := \{y \, : \, \exists\, \xi \in \mathcal{O}_{\epsilon,n}, \text{ such that } \xi_0 = x, \xi_n = y\}.$$

Starting at a point x, $\widehat{T}^n_\epsilon(x)$ is the set of points that may be reached in n steps by following the deterministic billiard but allowing for perturbations smaller than ϵ in the reflection angle.

Proposition 6. *If* $0 < l < n, n \geqslant 3$, $y \in \widehat{T}^n_\epsilon(x)$ *and there exists* z *in the interior of* M *such that* $z \in \widehat{T}^l_\epsilon(x)$ *and* $y \in \widehat{T}^{n-l}_\epsilon(z)$, *then* p^n_ϵ *is continuous at* (x, y) *and* $p^n_\epsilon(x, y) > 0$.

Proof. From the above definition we have that $\widehat{T}^n_\epsilon(x) = \bigcup_{z \in \widehat{T}^{n-1}_\epsilon(x)} \widehat{T}_\epsilon(z)$, and if $y \in \widehat{T}^2_\epsilon(x)$ and y do not belong to the boundary of M, then y belongs to the interior of the support of $p^2_\epsilon(x, .)$ and p^2_ϵ is continuous at (x, y). $\square$

Proposition 7. *Suppose that D is convex with C^1 boundary given by the union a finite number of C^2 arcs and line segments. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $x, y \in M$, there exists $\xi \in \mathcal{O}_{\epsilon,N}$, such that $\xi_0 = x$ and $\xi_N = y$.*

Proof. We split the proof in two steps.

First we show that it is possible to move between points in a given small neighborhood.

Finally we use this fact to cover the whole phase space.

Step 1. By definition, if $\theta_1 = \theta(Tx) \in [\epsilon, \pi - \epsilon]$, or equiv. $Tx \in M_\epsilon$, then $\widehat{T}_\epsilon x = \{s_1\} \times (\theta_1 - \epsilon, \theta_1 + \epsilon) \cap M$, and $\widehat{T}^2_\epsilon x$ is a distorted rectangle.

If $x \notin \partial M$, then $T^2 x$ lies in the interior of $\widehat{T}^2_\epsilon x$. Now, take $c_2(\epsilon) > 0$, fixed by the modulus of continuity of T as in Lemma 3. If $\delta < c_2$ is sufficiently small, then for all x in M_{c_2}, both $T^2(B_{2\delta}(x))$ and $B_{2\delta}(T^2 x)$ are contained in $\widehat{T}^2_\epsilon x$.

Consider a set $U \subset M_{c_1}$, $\nu(U) > 0$, diameter smaller than δ, and let $x_1 \in U$. As a consequence of the Poincaré Recurrence Theorem and B-Kh Ergodic Theorem: there exists z in U and $n_{_U} \leqslant (\nu(U))^{-1}$ with $T^{n_U}(z)$ in U.

By choice of δ, we have that $z_2 = T^2(z) \in \widehat{T}_\epsilon^2(x_1)$. From this we have that $T^{n_U-4}(z_2) = T^{n_U-2}(z)$ belongs to $\widehat{T}_\epsilon^{n_U-2}(x_1)$.

Note that, by the choice of c_2, since $T^{n_U}(z)$ belongs to M_{c_1}, then $T^{n_U-2}z$ belongs to M_{c_2} and so, again by the choice of δ, the ball of radius 2δ and center $T^2T^{n_U-2}(z)$ is contained $\widehat{T}_\epsilon^2(T^{n_U-2}z)$ and so $U \subset \widehat{T}_\epsilon^2(T^{n_U-2}z) \subset \widehat{T}_\epsilon^{n_U}x_1$.

Therefore our dynamics moves any point of U to any other point in U by the step $n_U \leqslant (\nu(U))^{-1}$.

Step 2. We partition the cylinder M_{c_1} into k rectangles $R_1, \dots, R_k$ based on a rectangular grid of size less then $\delta/2$, and consider the collection $Q_1, \dots, Q_l$ of rectangles of diameter less then δ, made of two adjacent rectangles R_i, R_j.

Let N_0 be such that N_0^{-1} is smaller that the minimum of $\nu(Q_j)$, $1 \leqslant j \leqslant l$. Then N_0 only depends on ϵ, and for each Q_j, there exists $n_{Q_j} < N_0$ such that, for any two points x, y in Q_j, y belongs to $\widehat{T}_\epsilon^{n_{Q_j}}(x)$. Let N_1 be the least common multiple of $\{1, \dots, N_0 - 1\}$.

Repeatedly applying same reasoning, any two points in the same rectangle can be joined by a random trajectory at step N_1: $\widehat{T}_\epsilon^{N_1} x$ contains Q_j for each $x \in Q_j$.

Consider two points x_0, y in M_{c_1}. There exists a sequence of adjacent rectangles $R_0, R_1, \ldots, R_m$, $m < k$, such that $x_0 \in R_0$ and $y \in R_m$. Choose $x_i \in R_i$, $1 \leqslant i \leqslant m-1$ and let $x_m = y$.

By construction, for any $0 \leqslant i \leqslant m-1$ there exists j_i such that both x_i and x_{i+1} belong to Q_{j_i}. Thus $x_{i+1} \in \widehat{T}_\epsilon^{N_1}(x_i)$.

By induction, $x_m \in \widehat{T}_\epsilon^{mN_1}(x_0)$. On the other hand, as a consequence of the recurrence of R_m by $\widehat{T}_\epsilon$, $x_m \in \widehat{T}_\epsilon^{N_1}(x_m)$ and so we have that $x_m \in \widehat{T}_\epsilon^{nN_1}(x_0)$ for any $n > m$.

Since x_0 and y were arbitrary and $m < k$, for any x in M_{c_1}, $\widehat{T}_\epsilon^{kN_1}(x)$ contains M_{c_1}. For any x in M, $\widehat{T}_\epsilon x$ intersects $M_\epsilon \subset M_{c_1}$, and so $\widehat{T}_\epsilon^{kN_1+1} x$ contains M_{c_1}.

If $\theta(Tx)$ is smaller than ϵ (greater than $\pi - \epsilon$) then $\widehat{T}_\epsilon x$ contains the segment $(s(Tx) \times [0, \epsilon])$ or the segment $(s(Tx) \times [\pi - \epsilon, \pi])$. Then, since $M_\epsilon \subset T(M_{c_1})$, $\widehat{T}_\epsilon(M_{c_1}) = M$ we have that $\widehat{T}_\epsilon^{kN_1+2} x = M$ for any $x \in M$. $\square$

Proof of Proposition 5. By Proposition 7, there exists $N \geqslant 2$ such that, for all $x, y \in M$, $y \in \widehat{T}_\epsilon^N(x)$. In particular, by Proposition 6, p_ϵ^N is continuous and strictly positive in (x, y). The result follows by compactness. □